11/7/2016

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2$$

$$\boxed{e^{2\beta} = \frac{1}{1 - \frac{2Gm(r)}{r}}}$$

$m(r) = \int 4\pi r^2 dr\, \rho(r)$

$G^t{}_t = 8\pi G\rho$ (5.143) (5.147)

$$\boxed{\alpha' = \frac{2G\left(m(r) + 4\pi r^3 p(r)\right)}{r\left(r - 2m(r)\right)}}$$

$\boxed{G^r{}_r = 8\pi G p}$ (5.152)

$T^{\mu\nu}{}_{;\nu} = 0$

$u^r = 0$

$$\alpha'(\rho + p) = \frac{\partial p}{\partial r}$$

$$\boxed{\frac{dp}{dr} = -\frac{(\rho + p)\left(Gm + 4\pi r^3 p\right)}{r\left(r - 2m\right)}}$$

Tolman–Oppenheimer–Volkoff. (5.154)

Static "star".

Need: $p = p(\rho)$

$p = k\rho^\gamma$

"polytrope"

②

Incompressible. $\rho = \text{constant} = \rho^*$.

$$m(r) = \frac{4\pi}{3}\rho^* r^3 = M\frac{r^3}{R^3} \qquad M = \frac{4\pi}{3}R^3$$

$$\frac{dp}{dr} = -\frac{(\rho^*+p)\left(\frac{4\pi}{3}r^3\right)(\rho^*+3p)}{r\left(r - \frac{2Mr^3}{R^3}\right)}$$

$$r^2\left(1 - \frac{2M}{R}\cdot\frac{r^2}{R^2}\right)$$

$$\times\frac{R^3\rho^*}{R^3\rho^*}$$

$$\frac{dp}{dr} = \frac{-Mr}{\rho^*}\frac{(\rho^*+p)(\rho^*+3p)}{R^3 - 2Mr^2}$$

Let: $x = R^3 - 2Mr^2$

$$\frac{dp}{dr} = \frac{dp}{dx}\frac{dx}{dr} = \frac{dp}{dx}(-4Mr)$$

$$\frac{dp}{dx} = +\frac{1}{4\rho^*}\frac{(\rho^*+p)(\rho^*+3p)}{x}.$$

$$\int_0^p \frac{dp\cdot\rho^*}{(\rho^*+p)(\rho^*+3p)} = \int_{R^3-2MR^2}^{R^3-2Mr^2}\frac{dX}{4X}.$$

$p=0$ at $r=R$

partial fractions,

$$\frac{\rho^*}{(\rho^*+p)(\rho^*+3p)} = \frac{A}{\rho^*+p} + \frac{B}{\rho^*+3p} = \frac{\rho^*(A+B)+p(A+3B)}{(\rho^*+p)(\rho^*+3p)}.$$

$A+B=1$ $\quad A+3B=0$ $\qquad \underline{A=-\tfrac{1}{2}} \quad \underline{B=\tfrac{3}{2}}$

$$\int_0^p dp \left(\frac{3/2}{\rho^*+3p} - \frac{1/2}{\rho^*+p} \right) = \frac{1}{2}\Big[\log(\rho^*+3p) - \log(\rho^*+p)\Big]_0^p$$

$$= \frac{1}{2}\log\left(\frac{\rho^*+3p}{\rho^*+p}\right) = \frac{1}{4}\log\left(\frac{R^3-2Mr^2}{R^3-2MR^2}\right)$$

$$\boxed{p = \rho^*\left(\frac{\sqrt{1-\frac{2Mr^2}{R^3}} - \sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}\right)}$$

(5.158)

(carroll. $\cancel{\alpha}<0$, $\cancel{\beta}<0$)

$$(\rho+p)\frac{d\alpha}{dr} = -\frac{dp}{dr}.$$

$r<R$

$$e^\alpha = \frac{3}{2}\left(1-\frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2GMr^2}{R^3}\right)^{1/2}$$

$r>R$ $\quad e^\alpha = \frac{3}{2}\left(1-\frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2GM}{R}\right)^{1/2} = \left(1-\frac{2GM}{R}\right)^{1/2}$

$(r=0)$ $\quad p = e^*\left(\dfrac{1-\sqrt{1-\frac{2m}{R}}}{3\sqrt{1-\frac{2m}{R}}-1}\right)$

Need: $3\sqrt{1-\frac{2m}{R}}-1 > 0.$

$$9\left(1-\frac{2m}{R}\right) > 1$$

$$1-\frac{2m}{R} > \frac{1}{9} \qquad \frac{2m}{R} < \frac{8}{9}.$$

$$\boxed{R > \frac{9}{8}\left(\frac{2Gm}{c^2}\right)}$$

Else: $p \to \infty$ can't support star

Dominant (~~Strong~~ energy) condition $\quad p_c \le e^*$

$$1-\sqrt{1-\frac{2m}{R}} \le 3\sqrt{1-\frac{2m}{R}}-1$$

$$2 \le 4\sqrt{1-\frac{2m}{R}}$$

$$1-\frac{2m}{R} > \frac{1}{4} \qquad \frac{2m}{R} < \frac{3}{4}$$

$$\boxed{R > \frac{4}{3}(2m)}$$

plot, $\frac{M}{R} = \frac{3}{8}$ $p_c = \rho^*$

green. $\frac{p(r)}{\rho^*}$ $p=0$ $r \geq R$

red. g_{tt}. $(-(1+\alpha))$ $\cancel{-(1-\frac{2m}{r})}$

blue g_{rr} $(1-\alpha)$. $\cancel{\frac{1}{(1-\frac{2m}{r})}}$

@ $r=0$ $e^{\alpha} = \frac{3}{2}\left(1-2\cdot\frac{3}{8}\right)^{1/2} - \frac{1}{2}\cdot 1$

$= \frac{3}{2}\left(\frac{8-6}{8}\right)^{1/2} - \frac{1}{2} = \frac{3}{2}\cdot\frac{1}{2} - \frac{1}{2} = +\frac{1}{4}$

$-e^{2\alpha} = -\frac{1}{16}$

main sequence. $L \propto M^{3.5}$ $n = 3.8$

$T \propto M^{-2.5}$

$R \propto M^{\xi}$ $\xi = 0.57$ $\cancel{>} M_\odot$ $0.8 < M_\odot$ (convective)

$$N = \int \frac{d^3x\, d^3p}{(2\pi\hbar)^3} = \frac{R^3 p^3}{\hbar^3} \qquad p = \frac{N^{1/3}\hbar}{R}$$

$$V = -\frac{3}{5}\frac{GM^2}{R} = -\frac{3}{5}\frac{GN^2 m_n^2}{R}$$

$$T = \frac{3}{5} \cdot \frac{N p^2}{2m}$$

virial theorem $\quad \frac{Np^2}{m} = \frac{GN^2 m_n^2}{R} = \frac{N}{m} \frac{N^{2/3}\hbar^2}{R^2}$

$$\boxed{R = N^{-1/3} \frac{\hbar c}{Gm_n^2} \cdot \frac{\hbar}{mc}}$$

Solar mass $\quad N = \frac{2 \times 10^{33}\, g}{1.67 \times 10^{-24}\, g} = 10^{57}$

$\frac{Gm_n^2}{\hbar c} = 10^{-38}$ $\quad 6 \cdot 10^{-39}$ $\qquad \frac{\hbar}{m_n c} = 2 \times 10^{-14}$ cm $\qquad 2 \cdot 10^{5}$ cm $\quad$ 3 km $\quad$ NS

$\frac{\hbar}{m_e c} = 4 \cdot 10^{-11}$ cm. $\qquad$ 6400 km $\quad$ WD.

$$p = \left(N^{1/3}\cancel{\hbar}\right)\left(N^{1/3}\frac{Gm_n^2}{\hbar c}\frac{mc}{\cancel{\hbar}}\right) = mc \qquad \boxed{N^{2/3} \cdot \frac{Gm_n^2}{\hbar c} = 1}$$

$$N = \left(\frac{\hbar c}{Gm_n^2}\right)^{3/2} = 10^{57}!$$