4/9/2016 **Gravitational waves.**

weak field. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

$$\rightarrow \bar{h}_{\mu\nu,\alpha}{}^{,\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{\nu} - \bar{h}_{\nu\alpha}{}^{,\alpha}{}_{\mu} = -16\pi G T_{\mu\nu}$$

(7.8)

gauge transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu} \qquad \| \qquad h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$$

(7.14)

$$\bar{h}'^{\mu\nu}{}_{;\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{,\nu}$$

$h_{\mu\nu} + \mathcal{L}_{\xi} g_{\mu\nu}$ (7.13)

choose Lorentz gauge. $\bar{h}^{\mu\nu}{}_{,\nu} = 0$.

$\rightarrow$ inhomogeneous $\xi^{\mu,\nu}{}_{,\nu}$ + any $\xi^{\mu,\nu}{}_{,\nu} = 0$

$$\Box^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Vacuum $\rightarrow T_{\mu\nu} = 0 \rightarrow \Box^2 \bar{h}_{\mu\nu} = 0$.

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \cdot e^{i(\vec{k}\cdot\vec{x} - \omega t)} = A_{\mu\nu} e^{i k_\alpha x^\alpha}.$$

$$\Box^2 \bar{h}_{\mu\nu} = \partial_\alpha \partial^\alpha \left(A_{\mu\nu} e^{i k_\alpha x^\alpha} \right) = -k_\alpha k^\alpha \bar{h}_{\mu\nu} = 0.$$

$k_\alpha k^\alpha = k \cdot k = 0$ $\quad \boxed{\omega^2 = |\vec{k}|^2}$ null.

gauge condition. $\bar{h}_{\mu\nu}{}^{,\nu} = 0 \rightarrow \boxed{A_{\alpha\beta} k^\beta = 0}$

$A_{(\alpha\beta)}$ (= 10) $\quad k^\alpha A_{\alpha\beta} = 0$. (−4) $\rightarrow$ (6) ✗

Still some gauge freedom: $\Box \xi^\mu = 0$

$$\xi^\mu = B^\mu e^{i k \cdot x}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \tfrac{1}{2} \eta_{\mu\nu} \left(2 \xi_\alpha{}^{,\alpha} \right)$$

$$A'_{\mu\nu} = A_{\mu\nu} - k_\mu B_\nu - k_\nu B_\mu + \eta_{\mu\nu} \left(k^\alpha B_\alpha \right)$$

Look at $A'_{\mu\nu} u^\nu$ $\quad u^\gamma$ = 4-velocity of observer = time.

$$A'_{\mu\nu} u^\nu = A_{\mu\nu} u^\nu - k_\mu B_\nu u^\nu - k_\nu u^\nu B_\mu + u_\mu k^\alpha B_\alpha = 0.$$

$$A'^\mu{}_\mu = A^\mu{}_\mu - k^\mu B_\mu - k^\mu B_\mu + 4 k^\mu B_\mu = 0$$

choose: $\boxed{k^\mu B_\mu = \tfrac{1}{2} A^\mu{}_\mu}$ $\quad A'^\mu{}_\mu = 0$

$$A'_{\mu 0} = A_{\mu 0} - k_\mu B_0 - k_0 B_\mu + (\eta_{\mu 0})\,(k^\alpha B_\alpha)$$

$\mu = 0$ $\quad A'_{00} = A_{00} - k_0 B_0 - k_0 B_0 + (-1)\left(\tfrac{1}{2} A^\mu{}_\mu\right) = 0.$

$\Rightarrow B_0.$

$(\mu = i)$ $\quad A'_{i0} = A_{i0} - k_i B_0 - k_0 B_i = 0$

$\rightarrow k_i \,, \quad \rightarrow B_i$

Conditions: $\quad k^\mu A_{\mu\nu} = 0 \quad u^\mu A_{\mu\nu} = 0 \quad A^\mu{}_\mu = 0$

"Transverse Traceless" (TT) gauge.

$| \; A_{\mu 0} = A_{0\mu} = 0 \qquad k = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}.$

$k^\mu A_{\mu\nu} = \omega\,(A_{0\mu} + A_{3\mu}) = 0$

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_+ \begin{pmatrix} & & & \\ & 1 & & \\ & & -1 & \\ & & & \end{pmatrix} \qquad A_\times = \begin{pmatrix} & & & \\ & & 1 & \\ & 1 & & \\ & & & \end{pmatrix}$$

Geodesic deviation

$$\frac{d^2 x^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma$$

(3.208)

$$\rightarrow \quad \frac{d^2 x^i}{dt^2} = - R^i{}_{0j0}\, \xi^j$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\mu,\beta\nu} + h_{\beta\nu,\alpha\mu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right)$$

$$R_{i0j0} = \frac{1}{2}\left(h_{i0,j0} + h_{j0,i0} - h_{ij,00} - h_{00,ij}\right)$$

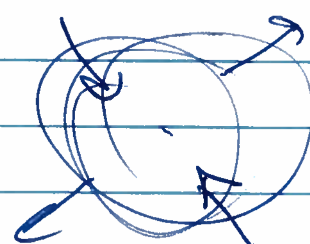
$$= +\omega^2 A_{ij}\, e^{ik\cdot x}$$

$$\xi^i = x_0^i + (\Delta x)^i e^{-i\omega t}$$

$$\frac{d^2 \xi^i}{dt^2} = -\omega^2 (\Delta x)^i = -\omega^2 A_{ij} (x_0)^j$$

$$(\Delta x)^i = A^i{}_j\, x_0^j$$

(+).



(×).

Rotation. $A'_{ij} = R_{im} R_{jn} A_{mn} = R A R^T$

$(+) \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Figure rotates w/ twice θ → "$s=2$"

$\theta = \frac{\pi}{4} \rightarrow \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \times$

Transverse: $P_{jk} = \delta_{jk} - \hat{n}_j \hat{n}_k$

(projects $\perp \hat{n}$). $P_{jk} P_{km} = P_{jm}$ $(P^2 = P)$

$h^T_{ij} = P_{im} h_{mn} P_{nj}$

trace $h^T_{ii} = P_{im} h_{mn} P_{ni} = P_{in} P_{im} h_{mn} = P_{mn} h_{mn}$

$h^{TT}_{ij} = P_{im} h_{mn} P_{nj} - P_{ij} P_{mn} h_{mn}$

$= PhP - P \cdot Tr(Ph)$

Gravitational stress-energy
"pseudo-tensor"

$h_{\mu\nu} \rightarrow R^{(1)}, R^{(2)}$

$$G_{\mu\nu} = G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(2)} = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu}^{(GW)} = -\frac{1}{8\pi G} G_{\mu\nu}^{(2)}(h)$$

$$T_{\mu\nu}^{(GW)} = \frac{1}{32\pi} \langle \bar{h}_{\alpha\beta,\mu} \bar{h}^{\alpha\beta}{}_{,\nu} - \frac{1}{2} \bar{h}_{,\mu} \bar{h}_{,\nu} - \bar{h}^{\alpha\beta}{}_{,\beta} \bar{h}_{\alpha\mu,\nu} - \bar{h}^{\alpha\beta}{}_{,\beta} \bar{h}_{\alpha\nu,\mu} \rangle$$

$$T_{\mu\nu}^{(GW)} = \frac{1}{32\pi} \langle h^{TT}_{ij,\mu} h^{TT}_{ij,\nu} \rangle$$

$\partial_\mu \rightarrow i k_\mu$

$$T_{00} = \frac{\omega^2}{32\pi} \langle \bar{h}_{\alpha\beta} \bar{h}^{*\alpha\beta} \rangle = \varepsilon$$

$$T_{0i} = \frac{\omega^2 \hat{k}_i}{32\pi} \langle \bar{h}_{\alpha\beta} \bar{h}^{*\alpha\beta} \rangle = \vec{S}_i$$

$$\frac{dP}{d\Omega} = r^2 \hat{r} \cdot \vec{S} = \frac{\omega^2}{32\pi} \langle r^2 \bar{h}^{TT}_{\alpha\beta} \bar{h}^{*TT\,\alpha\beta} \rangle$$