11/18/2016

Action formulation . $S = \int d^4x \, \mathcal{L}$

$d^4x' = \left|\det \frac{\partial x'^\mu}{\partial x^\nu}\right| d^4x$

$$\boxed{\sqrt{|g'|}\, d^4x' = \sqrt{|g|}\, d^4x}$$

$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$

$\Rightarrow \det g' = \left|\det \frac{\partial x^\alpha}{\partial x'^\mu}\right|^2 \det g$

$\sqrt{|g'|} = \left|\det \frac{\partial x^\alpha}{\partial x'^\mu}\right| \sqrt{|g|}$

Flat, cartesian $|g| = 1$.

$$\boxed{\text{proper volume} = \sqrt{|g|}\, d^4x}$$

$$\boxed{S = \int \sqrt{g}\, d^4x \, \mathcal{L}}$$

scalar

gravity . $g_{\mu\nu}$ $\quad R^\mu{}_{\nu\rho\sigma}$ $\quad R_{\mu\nu}$ $\quad G_{\mu\nu}$ $\quad C^\mu{}_{\nu\alpha\beta}$

$\partial^2 g$ $\quad (\partial g)^2$

No more derivatives $\rightarrow$ $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\sigma{}_{\mu\sigma\nu}$

$$\boxed{R = R^{\sigma\mu}{}_{\sigma\mu}}$$

Einstein-Hilbert action

$$S = \int \sqrt{g}\, d^4x\, R + (S_{matter}). = \int \sqrt{g} \cdot g^{\mu\nu} \cdot R_{\mu\nu}.$$

(—) ← normalization

$$\delta S = \int d^4x \cdot \left\{ \sqrt{g} \cdot g^{\mu\nu} \cdot \delta R_{\mu\nu} + \sqrt{g} \cdot R_{\mu\nu} \cdot \delta g^{\mu\nu} + R \cdot \delta(\sqrt{g}) \right\}$$

$$g^{\mu\lambda} \cdot g_{\lambda\nu} = \delta^{\mu}_{\nu} \rightarrow \delta g^{\mu\lambda} g_{\lambda\nu} + g^{\mu\lambda} \cdot \delta g_{\lambda\nu} = 0.$$

$$\boxed{\delta g_{\mu\nu} = - g_{\mu\rho}\, \delta g^{\rho\sigma}\, g_{\sigma\nu}} \qquad \delta\left(\frac{1}{x}\right) = -\frac{\delta x}{x^2}$$

As before: $\det g = \prod_i \lambda_i = \exp\left(\sum_i \log \lambda_i\right)$

$$\frac{\delta(\det g)}{(\det g)} = \frac{\delta g}{g} = \mathrm{Tr}\left(g^{-1} \delta g\right) = \mathrm{Tr}\left(\delta g\, g^{-1}\right)$$

$$\frac{\delta g}{g} = g^{\mu\nu} \cdot \delta g_{\mu\nu}$$

$$\frac{\delta(\sqrt{g})}{\sqrt{g}} = \frac{1}{2} g^{\mu\nu} \cdot \delta g_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \cdot \delta g^{\mu\nu}$$

$$\delta S = \int d^4x \cdot \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} + \int d^4x \left(\sqrt{-g}\, \delta g^{\mu\nu}\right)\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right)$$

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} + \cancel{(\Gamma^2)} \quad \text{Frame } (\Gamma = 0)$$

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda (\delta \Gamma^{\rho}_{\nu\mu}) - \partial_\nu (\delta \Gamma^{\rho}_{\lambda\mu})$$

$\Gamma^{\rho}_{\nu\mu}$ is not a tensor, but $\delta \Gamma^{\rho}_{\nu\mu}$ is $\quad \nabla' V - \nabla V$

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^{\rho}_{\nu\mu}) - \nabla_\nu (\delta \Gamma^{\rho}_{\lambda\mu})$$

Trace ($\rho\lambda$), $\quad \delta R_{\mu\nu} = \nabla_\lambda (\delta \Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu (\delta \Gamma^{\lambda}_{\lambda\mu})$

$$\delta S_1 = \int d^4x \sqrt{-g} \cdot g^{\mu\nu} \left[\nabla_\lambda (\delta \Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu (\delta \Gamma^{\lambda}_{\lambda\mu})\right]$$

$$\delta \Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} \left(\delta g_{\rho\mu,\nu} + \delta g_{\rho\nu,\mu} - \delta g_{\mu\nu,\rho}\right)$$

$$\boxed{\delta g_{\alpha\beta} = - g_{\alpha\mu}\, \delta g^{\mu\nu}\, g_{\nu\beta}}$$

$$\longrightarrow -\tfrac{1}{2} \left[g_{\lambda\mu} \nabla_\nu (\delta g^{\lambda\sigma}) + g_{\lambda\nu} \nabla_\mu (\delta g^{\lambda\sigma}) - g_{\mu\alpha} g_{\nu\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right]$$

$$\delta S_1 = \int d^4x \sqrt{-g} \cdot \nabla_\sigma \Big[\underbrace{g_{\mu\nu} \cdot \nabla^\sigma (\delta g^{\mu\nu}) - \nabla_\lambda (\delta g^{\sigma\lambda})}_{V^\sigma} \Big]$$

Useful formula

$$V^\sigma_{;\sigma} = V^\sigma_{,\sigma} + \Gamma^\sigma_{\lambda\sigma} V^\lambda$$

$$\boxed{V^\sigma_{;\sigma} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, V^\sigma \right)_{,\sigma}}$$

$$\Gamma^\sigma_{\lambda\sigma} = \frac{1}{2} g^{\sigma\rho} \left(\cancel{g_{\rho\lambda,\sigma}} + g_{\rho\sigma,\lambda} - \cancel{g_{\lambda\sigma,\rho}} \right)$$

$$= \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\lambda} = \partial_\lambda (\sqrt{-g})$$

$$\delta S_1 = \int d^4x \cdot \cancel{\sqrt{-g}} \left(\frac{1}{\cancel{\sqrt{-g}}} \left(\sqrt{-g} \cdot V^\sigma \right)_{,\sigma} \right)$$

$$= \int d^4x\, \partial_\sigma \left(\longrightarrow \right) \rightarrow \text{Surface} \rightarrow 0$$

$$\delta S_H = \int d^4x \sqrt{-g}\, \delta g^{\mu\nu} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right)$$

Vacuum . $\delta S_H = 0$. $G_{\mu\nu} = 0$

$$S = \frac{1}{16\pi G} S_H + S_{matter}$$

$\delta S = 0 \rightarrow$ $\frac{1}{16\pi G} G_{\mu\nu} + \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = 0$

$\delta S = 0 \rightarrow$ $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ $T_{\mu\nu} = -2 \frac{\delta S_m}{\delta g^{\mu\nu}}$

$$S_m = \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

$$\delta S_m = \int d^4x \sqrt{-g} \left(-\tfrac{1}{2} g_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \right)$$

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}$$

Automatically symmetric