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Einstein-Hilbert Action

$$S = \int d^4x \sqrt{g}\, R = \int d^4x \sqrt{g}\, g^{\mu\nu} R_{\mu\nu}$$

$$\delta S = \int d^4x \left\{ \sqrt{g}\, g^{\mu\nu} \delta R_{\mu\nu} + \underbrace{\sqrt{g}\, R_{\mu\nu} \delta g^{\mu\nu} + R_{\mu\nu} g^{\mu\nu} \delta(\sqrt{g})}_{\sqrt{g}\,\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)} \right\}$$

$\delta g^{\mu\nu} =$

$$\boxed{\frac{\delta(\sqrt{g})}{\sqrt{g}} = -\frac{1}{2} g_{\mu\nu}\, \delta g^{\mu\nu}}$$

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g}\, R + \int d^4x \sqrt{g}\, \mathcal{L}_{matter}$$

$$\boxed{T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}}$$

Automatically symmetric

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$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi).$$

$$T_{\mu\nu} = \nabla_\mu \phi \cdot \nabla_\nu \phi + g_{\mu\nu} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V \right)$$

$$T_{00} = (\dot{\phi})(\dot{\phi}) + (-1) \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} |\nabla \phi|^2 - V \right)$$

$$\boxed{T_{00} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + V.}$$

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{16\pi} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma}$$

$$(\sigma = \mu)(\rho = \nu) + (\rho = \mu)(\alpha = \nu) + (\beta = \mu)(\sigma = \nu) + (\sigma = \mu)(\beta = \nu)$$

$$T_{\mu\nu} = -\frac{1}{16\pi} g^{\beta\sigma} F_{\mu\beta} F_{\nu\sigma} (\times 4)$$

$$+ g_{\mu\nu} \left(\frac{-1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \right)$$

$$\boxed{T^{EM}_{\mu\nu} = \frac{1}{4\pi} F_\mu{}^\sigma F_{\nu\sigma} - \frac{1}{16\pi} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$

③

$$T^\mu{}_\nu = \begin{pmatrix} -\frac{E^2}{8\pi} & & & \\ & -\frac{E^2}{8\pi} & & \\ & & \frac{E^2}{8\pi} & \\ & & & \frac{E^2}{8\pi} \end{pmatrix}$$

$\frac{\partial}{\partial t} = 0$ $\frac{\partial}{\partial \theta} = 0$ $\frac{\partial}{\partial \phi} = 0$ $\vec{E} = E_r \hat{r}$ $B = 0$

$T^\mu{}_\mu = 0 \rightarrow G^\mu{}_\mu = 0 \rightarrow R^\mu{}_\nu = G^\mu{}_\nu$

$$R^r{}_r = -e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' - \frac{2\beta'}{r}\right) = \left(-E^2\right)$$

$$G^\theta{}_\theta = G^\phi{}_\phi = e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha' - \beta'}{r}\right) = \left(+E^2\right)$$

$$R^r{}_r + G^\theta{}_\theta = 0 = \frac{e^{-2\beta}}{r}(\alpha' + \beta') \qquad \alpha = -\beta$$

$$ds^2 = -e^{2\alpha}dt^2 + e^{-2\alpha}dr^2 + r^2 d\Omega^2$$

$$g = -e^{2\alpha} \cdot e^{-2\alpha} \cdot r^2 \cdot r^2 \sin^2\theta = -(r^2 \sin\theta)^2$$

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$$F^{\mu\nu} = -F^{\nu\mu}$$

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,F^{\mu\nu}\right)_{,\nu}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 0 \qquad E_r = \frac{Q}{r^2}$$

Relativistically exact.

$$A_p = 4\pi r^2$$

$$r\left(1-e^{-2B}\right) = 2m$$

$$1-e^{-2B} + 2rB'e^{-2B} = 2m' = 8\pi r^2\rho$$

$$m = m_0 + \int 4\pi r^2\,dr\left(\frac{Q^2/r^4}{8\pi}\right) = m_0 - \frac{Q^2}{2r}$$

$$\frac{dm}{dr} = +\frac{Q^2}{2r^2} = +4\pi r^2\,\frac{(Q^2/r^2)}{8\pi}$$

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}+\frac{Q^2}{r^2}} + r^2 d\Omega^2$$

$$|Q| < M$$

Timelike ξ^μ

$$\underbrace{J^\mu = T^{\mu\nu}\xi_\nu}$$

$$D_\mu J^\mu = (T^{\mu\nu}\xi_\nu)_{;\mu} = \cancel{T^{\mu\nu}{}_{;\mu}}\,\xi_\nu + T^{\mu\nu}\cancel{\xi_{\nu;\mu}} = 0$$

Global conserved "Energy"

$$E = \int d^3\Sigma_\mu \cdot T^{\mu\nu}\xi_\nu$$

(Σ) spacelike hypersurface.