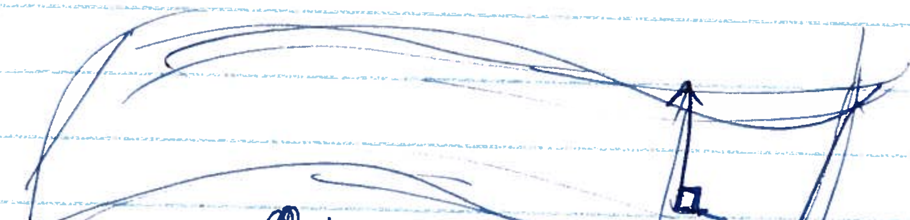
12/2/2016 (manifestly covariant) ${}^{(4)}g_{\mu\nu} \to \underline{3+1}$.



Σ = spacelike hypersurface

$\underline{\hat{e}_i}$ in Σ ${}^{(3)}ds^2 = g_{ij}\,dx^i dx^j$ restricted to Σ.

Normal (4d). $\hat{n}$ $\hat{n}\cdot\hat{n} = -1$
$\hat{n}\cdot\hat{e}_i = 0$.

$${}^{(4)}\nabla_i \hat{e}_j = {}^{(4)}\Gamma^{\mu}_{ji}\,\hat{e}_\mu$$

$$\hat{n}\cdot\left({}^{(4)}\nabla_i \hat{e}_j\right) = {}^{(4)}\Gamma^{\mu}_{ji}\,(\hat{n}\cdot\hat{e}_\mu) = \underline{{}^{(4)}\Gamma^{0}_{ji}}\,(\hat{n}\cdot\hat{e}_0)$$

project ⊥ Σ.

$\underline{{}^{(4)}\nabla_i \hat{n} = ?}$ $\hat{n}\cdot\hat{n} = -1$ → no change along its length.

$$\to \boxed{{}^{(4)}\nabla_i \hat{n} = -K_i^{\ j}\,\hat{e}_j}$$

"Extrinsic curvature".

$\underline{\text{converging}} \to \underline{K>0}$

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$\hat{e}_k \cdot \rightarrow \; ({}^{(4)}\nabla_i \hat{n}) \cdot \hat{e}_k = -K_i{}^j g_{jk}.$

$$\boxed{K_{ik} = -n_{i;k}}$$

$\hat{n} \cdot \hat{e}_j = 0$

$(\nabla \hat{n}) \cdot \hat{e} + \hat{n} \cdot (\nabla \hat{e}) = 0$

$$\underline{K_{ij}} = K_i{}^k g_{kj} = K_i{}^k (\hat{e}_k \cdot \hat{e}_j)$$

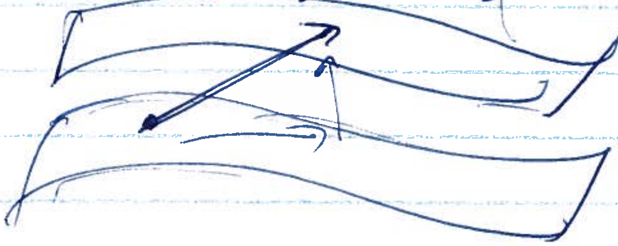
$$= (K_i{}^k \hat{e}_k) \cdot \hat{e}_j = -({}^{(4)}\nabla_i \hat{n}) \cdot \hat{e}_j$$

$$= +\hat{n} \cdot ({}^{(4)}\nabla_i \hat{e}_j) = \hat{n} \cdot ({}^{(4)}\Gamma^\mu_{ji} \hat{e}_\mu)$$

$$= (\hat{n} \cdot \hat{e}_0)\, {}^{(4)}\Gamma^0_{ij} = K_{ji}$$

Symmetric

Slice into stack of Σ_j.



Shift. $N^i dt$

Lapse N

(choose later)

$$\left.\begin{array}{l}\text{Shift } N^i dt \\ \text{Lapse } N\end{array}\right\} \quad ds^2 = g_{ij}\,(x^i + N^i dt)(x^j + N^j dt) - N^2 dt^2$$

$$ds^2 = -dt^2\left(N^2 - g_{ij}N^iN^j\right) + 2\, g_{ij} N^i dx^j dt + g_{ij}\, dx^i dx^j$$

$$^{(4)}g_{00} = -N^2 + N_j N^j$$

$$^{(4)}g_{0i} = {}^{(4)}g_{i0} = N_i$$

$$^{(4)}g_{ij} = {}^{(3)}g_{ij}$$

$$N_i = g_{ij}N^j \qquad \boxed{N^i = g^{ij} N_j}$$

$^{(4)}g^{\mu\nu}$ = matrix inverse of $^{(4)}g_{\mu\nu}$.

$$^{(4)}g^{00} = -1/N^2$$

$$^{(4)}g^{0i} = {}^{(4)}g^{i0} = N^i/N^2$$

$$^{(4)}g^{ij} = \left(g^{ij} - \frac{N^iN^j}{N^2}\right) \neq {}^{(3)}g^{ij}$$

$$g^{ij} - \frac{N^iN^j}{N^2}$$

Check. $^{(4)}g^{0\mu}\,{}^{(4)}g_{\mu 0} = {}^{(4)}g^{00}\,{}^{(4)}g_{00} + {}^{(4)}g^{0i}\,{}^{(4)}g_{0i}$

$$= \left(-\frac{1}{N^2}\right)\left(-N^2 + \cancel{N_j N^j}\right) + \cancel{\left(\frac{N^i}{N^2}\right)\left(N_i\right)} = 1 \quad \checkmark$$

normal. $\underline{\hat{n}\cdot\hat{n} = -1}$ $\quad \underline{\hat{n}\cdot\hat{e}_j = 0}$

$\underline{n_\mu = (n_0, 0)}$

$n^0 = {}^{(4)}g^{00} n_0 = -\frac{1}{N^2} n_0$

$\hat{n}\cdot\hat{n} = (n_0)\left(-\frac{1}{N^2} n_0\right) = -1 \qquad n_0 = -N$

$n^0 = \frac{1}{N}$

$n^i = {}^{(4)}g^{i0} n_0 = -\frac{N^i}{N}$

$n^\mu = \frac{1}{N}(1, -N^i)$ ← undo shift.

scale to proper length

$^{(4)}g^{i\mu}\,{}^{(4)}g_{\mu j} = {}^{(4)}g^{i0}\,{}^{(4)}g_{0j} + {}^{(4)}g^{ik}\,{}^{(4)}g_{kj}$

$= \left(\frac{N^i}{N^2}\right)(N_j) + \left(g^{ik} - \frac{N^i N^k}{N^2}\right)(g_{kj}) = \delta^i_j - \frac{N^i N_j}{N^2}$

$$dx^{\mu}_{\perp} = \frac{1}{N}\left(dt, -N^i dt\right)$$

$$ds^2 = \left(-N^2 + N_i N^i\right)\left(\frac{dt}{N}\right)^2 + 2 N_i \left(-N^i dt\right) + g_{ij}\left(N^i dt\right)\left(N^j dt\right)$$

$$= -dt^2$$

There are 3d, 4d covariant derivatives

$${}^{(4)}\nabla_i \hat{e}_j = {}^{(4)}\Gamma^{\mu}_{ji}\, \hat{e}_{\mu} \quad \leftarrow \text{(includes } \hat{e}_0\text{)}$$

$${}^{(3)}\nabla_i \hat{e}_j = {}^{(3)}\Gamma^{k}_{ji}\, \hat{e}_k . \qquad \boxed{A^i{}_{|j} = A^i{}_{,j} + \Gamma^i_{kj} A^k}$$

$${}^{(4)}\Gamma^{\mu}_{jk} = \frac{1}{2}\, {}^{(4)}g^{\mu\nu}\left({}^{(4)}g_{\nu j,k} + {}^{(4)}g_{\nu k,j} - {}^{(4)}g_{jk,\nu}\right)$$

$$K_{ij} = \left(\hat{n}\cdot\hat{e}_0\right) {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= (-N)\left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0k}\, {}^{(4)}\Gamma_{kij}\right)$$

(6)

$$K_{ij} = \frac{1}{N}\left({}^{(4)}\Gamma_{0ij} - N^k \, {}^{(4)}\Gamma_{kij}\right)$$

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$$K_{ij} = \frac{1}{N}\cdot\frac{1}{2}\left(\frac{\partial N_i}{\partial x_j} + \frac{\partial N_j}{\partial x_i} - \frac{\partial g_{ij}}{\partial t} - 2N^k \Gamma_{kij}\right)$$

$$\boxed{K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \frac{\partial g_{ij}}{\partial t}\right)}$$

Ex: $ds^2 = a^2(t)\left(d\chi^2 + \sin^2\chi \, d\Omega^2\right)$ (3-sphere)

take $\underline{N^i = 0}$ $N =$ constant (χ, θ, ϕ). $\underline{N(t)} = 1$.

$$n^\mu = (1, \vec{0})$$

$$\boxed{K^i{}_j = -\frac{1}{2}\,\frac{1}{a^2}\frac{d}{dt}(a^2)\cdot\delta^i{}_j}$$

$$= -\left(\frac{\dot{a}}{a}\right)\delta^i{}_j$$