12/5/2016 $\quad {}^{(4)}\nabla_i \hat{n} = -K_i{}^j \hat{e}_j$ $\quad$ extrinsic curvature

$K_{ij} = (\hat{n}\cdot\hat{e}_0)\, {}^{(4)}\Gamma^0_{ij}$ $\quad$ symmetric $\quad$ choose $\hat{e}_0 \to \hat{n}$

$\mu = n$ $\quad$ $\mu = m$

$${}^{(4)}\nabla_j\, {}^{(4)}\nabla_k \hat{e}_i = {}^{(4)}\nabla_j \left(K_{ik} \frac{\hat{n}}{(\hat{n}\cdot\hat{n})} + {}^{(3)}\Gamma^m_{ik}\, \hat{e}_m \right)$$

Γ

$$= K_{ik,j} \frac{\hat{n}}{\hat{n}\cdot\hat{n}} + \frac{K_{ik}}{\hat{n}\cdot\hat{n}}\, {}^{(4)}\nabla_j \hat{n}$$

$$+ {}^{(3)}\Gamma^m_{ik,j}\, \hat{e}_m + {}^{(3)}\Gamma^m_{ik}\, {}^{(4)}\nabla_j \hat{e}_m$$

$$= K_{ik,j} \frac{\hat{n}}{\hat{n}\cdot\hat{n}} + \frac{K_{ik}}{\hat{n}\cdot\hat{n}} \left(-K_j{}^m \hat{e}_m \right)$$

$$+ {}^{(3)}\Gamma^m_{ik,j}\, \hat{e}_m + {}^{(3)}\Gamma^m_{ik} \left(\frac{K_{mj}\, \hat{n}}{\hat{n}\cdot\hat{n}} + {}^{(3)}\Gamma^p_{mj}\, \hat{e}_p \right)$$

$$= \frac{\hat{n}}{\hat{n}\cdot\hat{n}} \left(K_{ik,j} + {}^{(3)}\Gamma^m_{ik} K_{mj} \right)$$

$$+ \hat{e}_m \left(-\frac{K_{ik} K_j{}^m}{\hat{n}\cdot\hat{n}} + {}^{(3)}\Gamma^m_{ik,j} + {}^{(3)}\Gamma^p_{ik}\, \Gamma^m_{pj} \right)$$

②

$$\nabla_j \nabla_k \hat{e}_i - \nabla_k \nabla_j \hat{e}_i$$

$$= \frac{\hat{n}}{\hat{n}\cdot\hat{n}} \left(K_{ik,j} + {}^{(3)}\Gamma^m_{ik} K_{mj} - K_{ij,k} - {}^{(3)}\Gamma^m_{ij} K_{mk} \right)$$

$$+ \hat{e}_m \left(\frac{1}{\hat{n}\cdot\hat{n}} \left(K_{ij} K_k{}^m - K_{ik} K_j{}^m \right) - {}^{(3)}\Gamma^m_{ij,k} + {}^{(3)}\Gamma^m_{ik,j} + \Gamma\Gamma - \Gamma\Gamma \right)$$

$$K_{ik}|_j = K_{ik,j} - \Gamma^m_{ij} K_{mk} - \cancel{\Gamma^m_{kj} K_{im}}$$

$$K_{ij}|_k = K_{ij,k} - \Gamma^m_{ik} K_{mj} - \cancel{\Gamma^m_{jk} K_{im}}$$

$${}^{(4)}\nabla_j {}^{(4)}\nabla_k \hat{e}_i - {}^{(4)}\nabla_k {}^{(4)}\nabla_j \hat{e}_i$$

$$= \frac{\hat{n}}{(\hat{n}\cdot\hat{n})} \left(K_{ik}|_j - K_{ij}|_k \right)$$

$$+ \hat{e}_m \left[\frac{1}{\hat{n}\cdot\hat{n}} \left(K_{ij} K_k{}^m - K_{ik} K_j{}^m \right) + {}^{(3)}R^m{}_{ijk} \right.$$

3

$$^{(4)}R^{n}{}_{ijk} = -\frac{1}{\hat{n}\cdot\hat{n}}\left(K_{ij}\|_{k} - K_{ik}|_{j}\right)$$

$$= \left(\frac{1}{\hat{n}\cdot\hat{n}}\right) {}^{(4)}R_{nijk}$$

$$^{(4)}R^{m}{}_{ijk} = {}^{(3)}R^{m}{}_{ijk} + \frac{1}{\hat{n}\cdot\hat{n}}\left(K_{ij}K_{k}{}^{m} - K_{ik}K_{j}{}^{m}\right)$$

Gauss–Codazzi equations

Gaussian normal coordinates

Propagate along geodesics starting along $\hat{n}$, preserve $\hat{n}\cdot\hat{n}$

$$^{(4)}R^{n}{}_{ink} = (\hat{n}\cdot\hat{n})^{-1}\left(\frac{\partial K_{ik}}{\partial n} + K_{im}K^{m}{}_{k}\right)$$

$$R_{nink} = \frac{\partial}{\partial t}K_{ik} + K_{im}K^{m}{}_{k}$$

GNC only.

4

$$^{(4)}G^n{}_n = {}^{(4)}R^n{}_n - \tfrac{1}{2}\,{}^{(4)}R$$

$$(K)^2 - (K)(K)$$

$$= -\tfrac{1}{2}\,{}^{(3)}R + \tfrac{1}{2}(\hat n\cdot\hat n)\left(K_i{}^i K_j{}^j - K_{im}K^{im}\right)$$

$$^{(4)}G^n{}_j = {}^{(4)}R^n{}_j = (\hat n\cdot\hat n)\left(K_{|j} - K^i{}_{j|i}\right)$$

$$^{(4)}G^i{}_j = {}^{(4)}R^i{}_j - \tfrac{1}{2}\delta^i_j\,{}^{(4)}R$$

$$= {}^{(3)}G^i{}_j + (\hat n\cdot\hat n)\Big[K^i{}_{j,n} - K K^i{}_j - \tfrac{1}{2}\delta^i_j\left(2K_{,n} - K_{im}K^{im} - K^2\right)\Big]$$

Cosmology. $K_i{}^i = -\frac{3\dot a}{a}$ $K_i{}^j K^i{}_j = \left(\frac{\dot a}{a}\right)^2$

$K_i{}^j = -\left(\frac{\dot a}{a}\right)\delta_i{}^j$ $G^n{}_n = \left(-\frac{1}{2}\right)\left(\frac{3}{a^2}\right) - \frac{1}{2}\left(\left(\frac{9\dot a}{a}\right)^2 - 3\left(\frac{\dot a}{a}\right)^2\right)$

$-\frac{3}{2a^2} - 3\left(\frac{\dot a}{a}\right)^2 = -8\pi G\rho$ $\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}}$

$\boxed{G^i{}_j = 8\pi G p\,\delta^i{}_j} \rightarrow \boxed{\left(\frac{\ddot a}{a}\right) = -\frac{4\pi}{3}G(\rho + 3p)}$ kinematic

$\boxed{\dot K^i{}_j = \left(-\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2\right)\delta^i{}_j}$ dynamic recall: $ds^2 = -N^2 dt^2 + 2N_i\, dx^i dt + g_{ij}\, dx^i dx^j$

$(-\gamma^{ij} \beta_i)$

normal $N_i = 0$ $N = 1$ $\rightarrow \underbrace{-dt^2}_{d\omega^2} + g_{ij}\, dx^i dx^j$

$$K_{ik} = -n_{i;k} = \frac{1}{2\cancel{N}}\left(\cancel{N_{i|k}} + \cancel{N_{k|i}} - \frac{\partial g_{ik}}{\partial t}\right)$$

$$K_{ik} = -\frac{1}{2}\frac{\partial g_{ik}}{\partial t} \qquad K_{ik} = -\frac{1}{2}\mathcal{L}_m g_{ik}$$

$$S = \int d^4x \sqrt{g}\, \frac{{}^{(4)}R}{16\pi}$$

$${}^{(4)}R = R^\alpha{}_\alpha = R^{\alpha\beta}{}_{\alpha\beta} \qquad \beta = n, j$$

$$= R^{\alpha n}{}_{\alpha n} + R^{\alpha j}{}_{\alpha j} \qquad \alpha = n, i$$

$$= \cancel{R^{nn}{}_{nn}} + \underline{R^{in}{}_{in}} + \underline{R^{nj}{}_{nj}} + R^{ij}{}_{ij}$$

$$= 2\, {}^{(4)}R^{in}{}_{in} + {}^{(4)}R^{ij}{}_{ij}$$

$$= \underline{2\, {}^{(4)}R^{in}{}_{ij}} + \left[{}^{(3)}R^{ij}_{ij} - K^j{}_i K_j{}^i + K^j{}_j K_i{}^i\right)$$

Tear show: divergence $+ \underline{KK}$ terms.

$$S = \int d^4x \sqrt{g}\, \frac{1}{16\pi}\left({}^{(3)}R - K_i{}^i K_j{}^j + K_i{}^j K_j{}^i \right)$$

ADM

Field momentum $\pi^{ij} = \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}}$

$$= \sqrt{g}\,\left(g^{ij} K^k{}_k - K^{ij} \right)$$

w/ N, N^i

$$16\pi \mathcal{L} = -g_{ij}\dot{\pi}^{ij} - N\mathcal{H} - N_i \mathcal{H}^i$$

$$-2\left(\pi^{ij} N_j - \tfrac{1}{2} N^i \pi^j_j + N^{|i}\sqrt{g}\right)_{,i}$$

mess, surface terms, other fields

$$S = \int d^4x \sqrt{g}\, \left(\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N_i \mathcal{H}^i \right)$$

$$\mathcal{H} = g^{-1/2}\left(\pi^i{}_j \pi^j{}_i - \tfrac{1}{2}\pi^i{}_i \pi^j{}_j \right) - g^{1/2} R$$

$\uparrow$ super-Hamiltonian

$$\mathcal{H}^i = -2\pi^{ik}{}_{|k}$$ $\leftarrow$ super momentum

$\delta N \to \mathcal{H} = 0$

$\delta N_i \to \mathcal{H}^i = 0$

$\}$ constraints

Quantize.

covariant. $g_{\mu\nu}$ --

Canonical $[g_{ij}(x), \pi^{ij}(x')] = \delta(x-x')$