

## Random-matrix theory and the cumulants of the conductance for quasi-one-dimensional mesoscopic systems

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The behavior of the cumulants of the conductance of a quasi-one-dimensional mesoscopic system is analyzed, as a function of the number of channels  $N$  and the length  $s = L/\ell$ , within the random-matrix diffusion equation description. The study is performed for the case of unitary symmetry ( $\beta = 2$ ). It is shown that for  $N \gg 1$  the  $l$ th order cumulant  $\kappa_l$  can be expanded as a series in decreasing powers of  $N$  starting with  $N^{2-l}$ , and that, for  $s \gg 1$ , the term  $N^k$  contains  $1/s^k$  in leading order. Therefore, in the limit  $N \rightarrow \infty$ ,  $s \rightarrow \infty$ , with  $g_0 = N/s$  fixed,  $\kappa_l$  is given as an expansion in decreasing powers of  $g_0$ , starting with  $g_0^{2-l}$ . As  $g_0 \rightarrow \infty$ , the distribution of the conductance becomes Gaussian; if  $g_0 > 1$  and finite, there are corrections to the Gaussian behavior.

### I. INTRODUCTION

The statistical distribution of the conductance of a mesoscopic system has been investigated by a number of authors. In Ref. 1, diagrammatic methods are employed to find that the  $n$ th cumulant of the dimensionless conductance  $g$  [ $g = G/(e^2/h)$ ] of a cube of side  $L$  behaves, in the metallic regime, as

$$\kappa_n \sim (\bar{g})^{2-n}, \quad (1.1)$$

for  $n \leq n_0 \sim \ln(L/\ell)$ ,  $\ell$  being the elastic mean free path. The behavior of higher-order cumulants, giving rise to a log-normal tail of the distribution, is also studied.

A random-matrix theory (RMT) approach was employed in Refs. 2-4 to study conductance fluctuations by proposing a global maximum entropy ansatz for the full system. The RMT local approach of Refs. 5 and 6 starts from a statistical assumption for the scattering from successive slices of a quasi-one-dimensional system and finds a diffusion equation for the evolution, with increasing length, of the probability distribution of the transfer-matrix parameters; that approach was employed in Refs. 7-9 to calculate the variance of the conductance and correlation coefficients, giving results in agreement with diagrammatic and numerical calculations for quasi-one-dimensional systems; the calculation of higher-order cumulants was not attempted in those references. The diffusion equation for the unitary case<sup>8</sup> ( $\beta = 2$ ) was solved exactly in Ref. 10, while Ref. 11 solves it for arbitrary symmetry ( $\beta = 1, 2, 4$ ) using a linearization procedure: Those solutions could be used, in principle, to study the conductance distribution. An analysis of the first three cumulants arising from the diffusion equation is given in Ref. 12.

The RMT approach has the distinctive feature of being nonperturbative and of providing a model that contains those parameters that are believed to be physically rel-

evant. The purpose of the present paper is to analyze, within the random-matrix diffusion equation description of quasi-one-dimensional systems described above, the behavior of cumulants of the conductance of *arbitrary order*, for the case of unitary symmetry ( $\beta = 2$ ). We study the metallic regime and setup an expansion in inverse powers of the number of channels  $N$ , just as was done in Refs. 7 and 8 for the variance.

The paper is organized as follows. In the next section we find the evolution equation for the cumulant generating function (CGF). In Sec. III, the CGF is expanded in a series in decreasing powers of  $N$  [Eq. (3.1)] and a recursion scheme is generated; one of the central results of the paper, establishing the structure of the various terms occurring in the expansion of the CGF, is given in the statement associated with Eq. (3.21). A similar expansion for the various cumulants, Eq. (3.24), is then found. The length dependence is analyzed in Sec. IV; the second important result of the paper is Eq. (4.36), which gives the asymptotic form of the CGF terms in the metallic regime. A summary and conclusions are given in Sec. V.

### II. EVOLUTION OF THE CUMULANT GENERATING FUNCTION

Let  $t = \{t_{ab}\}$  be the matrix of transmission amplitudes ( $a, b = 1, \dots, N$  label channels) for our sample and  $\tau_a$  ( $a = 1, \dots, N$ ) the eigenvalues of the Hermitian matrix

$$h = tt^\dagger. \quad (2.1)$$

The total transmission coefficient is

$$T = \text{tr}(tt^\dagger) = \sum_{a=1}^N \tau_a. \quad (2.2)$$

The joint probability density  $w(\boldsymbol{\lambda})$  of the quantities  $\lambda_a \geq 0$ , related to the eigenvalues  $\tau_a$  through

$$\tau_a = \frac{1}{1 + \lambda_a}, \tag{2.3}$$

is found in Refs. 5, 6, and 8 for  $\beta = 1, 2$  and in Ref. 13 for  $\beta = 4$  to evolve with the length of the system according to the diffusion equation

$$\begin{aligned} \partial_s w_s(\lambda) = & \frac{2}{\beta N + 2 - \beta} \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} \\ & \times \left[ \lambda_a(1 + \lambda_a) J_\beta(\lambda) \frac{\partial}{\partial \lambda_a} \frac{w_s(\lambda)}{J_\beta(\lambda)} \right]. \end{aligned} \tag{2.4}$$

Here,  $s$  is the length of the system  $L$  measured in units of the mean free path  $\ell$ ,

$$s = L/\ell, \tag{2.5}$$

while  $J_\beta(\lambda)$  is the Jacobian

$$J_\beta(\lambda) = \prod_{a \leq b} |\lambda_a - \lambda_b|^\beta. \tag{2.6}$$

The initial condition appropriate to (2.4) is the one-sided  $\delta$  function

$$w_0(\lambda) = \delta(\lambda). \tag{2.7}$$

The parameter  $\beta$  takes on the values  $\beta = 1, 2, 4$ , for the orthogonal, unitary, and symplectic symmetry, respectively. We consider, in what follows, the unitary case  $\beta = 2$ . The expectation value of the function  $F(\lambda)$  evolves according to the equation<sup>8</sup>

$$\begin{aligned} N \partial_s \langle F \rangle_s = & \left\langle \sum_a \left[ \lambda_a(1 + \lambda_a) \frac{\partial^2 F}{\partial \lambda_a^2} + (1 + 2\lambda_a) \frac{\partial F}{\partial \lambda_a} \right] \right. \\ & + \sum_{a \neq b} \frac{1}{\lambda_a - \lambda_b} \left[ \lambda_a(1 + \lambda_a) \frac{\partial F}{\partial \lambda_a} \right. \\ & \left. \left. - \lambda_b(1 + \lambda_b) \frac{\partial F}{\partial \lambda_b} \right] \right\rangle_s, \end{aligned} \tag{2.8}$$

where  $\langle \dots \rangle_s$  indicates an expectation value calculated with the joint probability density  $w_s(\lambda)$ .

As an example, choose  $F = T^p$ ; we find<sup>8</sup>

$$N \partial_s \langle T^p \rangle_s = \langle -p T^{p+1} + p(p-1) T^{p-2} (T_2 - T_3) \rangle_s, \tag{2.9}$$

where

$$T_k = \sum_{a=1}^N \tau_a^k. \tag{2.10}$$

The  $T^p$ 's ( $p = 1, 2, \dots$ ) do not form a "closed set," in the sense that the new quantities  $T_k$  appear on the right-hand side of (2.9). On the other hand, we can convince ourselves that the set of *all* monomials of the form

$$\prod_k T_k^{p_k} \tag{2.11}$$

is a closed set, thus, *all* of the quantities  $T_k$  of Eq. (2.10), for  $k = 1, 2, \dots$ , must be considered simultaneously. For this purpose, we shall find it advantageous to construct the *moment generating function* (MGF) for the variables  $T_k$  ( $k = 1, 2, \dots$ ), defined as<sup>14</sup>

$$\begin{aligned} Z_s(\mathbf{q}) = & \left\langle \exp \left( - \sum_{k=1}^{\infty} q_k T_k \right) \right\rangle_s \\ = & \langle \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s \end{aligned} \tag{2.12}$$

[where  $\mathbf{q} = (q_1, q_2, \dots)$ ,  $\mathbf{T} = (T_1, T_2, \dots)$ ], as well as the (CGF), defined as<sup>14</sup>

$$\varphi_s(\mathbf{q}) = \ln Z_s(\mathbf{q}). \tag{2.13}$$

Application of the evolution equation (2.8) gives

$$\begin{aligned} N \partial_s \langle \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s = & \left\langle \left\{ \sum_{k=1}^{\infty} k q_k \left[ \sum_{l=0}^{k-1} T_{k-l} T_{l+1} \right. \right. \right. \\ & \left. \left. - \sum_{l=0}^{k-2} T_{k-l-1} T_{l+1} \right] \right. \\ & \left. + \sum_{k,l=1}^{\infty} k l q_k q_l (T_{k+l} - T_{k+l+1}) \right\} \\ & \times \exp(-\mathbf{q} \cdot \mathbf{T}) \Big\rangle_s. \end{aligned} \tag{2.14}$$

Since we can write

$$\langle T_k \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s = - \frac{\partial Z_s(\mathbf{q})}{\partial q_k}, \tag{2.15}$$

$$\langle T_k T_l \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s = \frac{\partial^2 Z_s(\mathbf{q})}{\partial q_k \partial q_l}, \tag{2.16}$$

we find, for the MGF, the evolution equation

$$\begin{aligned} N \partial_s Z_s(\mathbf{q}) = & \sum_{k=1}^{\infty} k q_k \left[ \sum_{l=0}^{k-1} \frac{\partial^2 Z_s(\mathbf{q})}{\partial q_{k-l} \partial q_{l+1}} \right. \\ & \left. - \sum_{l=0}^{k-2} \frac{\partial^2 Z_s(\mathbf{q})}{\partial q_{k-l-1} \partial q_{l+1}} \right] \\ & + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial Z_s(\mathbf{q})}{\partial q_{k+l+1}} - \frac{\partial Z_s(\mathbf{q})}{\partial q_{k+l}} \right]. \end{aligned} \tag{2.17}$$

Dividing (2.15) and (2.16) by  $Z_s(\mathbf{q})$ , we have

$$\frac{\langle T_k \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s}{Z_s(\mathbf{q})} = - \frac{\partial \varphi_s(\mathbf{q})}{\partial q_k}, \tag{2.18}$$

$$\frac{\langle T_k T_l \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s}{Z_s(\mathbf{q})} = \frac{\partial^2 \varphi_s(\mathbf{q})}{\partial q_k \partial q_l} + \frac{\partial \varphi_s(\mathbf{q})}{\partial q_k} \frac{\partial \varphi_s(\mathbf{q})}{\partial q_l}, \tag{2.19}$$

so that we find, for the CGF, the evolution equation

$$\begin{aligned}
N\partial_s\varphi_s(\mathbf{q}) &= \sum_{k=1}^{\infty} kq_k \left\{ \sum_{l=0}^{k-1} \left[ \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{k-l}} \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{l+1}} \right. \right. \\
&\quad \left. \left. + \frac{\partial^2\varphi_s(\mathbf{q})}{\partial q_{k-l}\partial q_{l+1}} \right] \right. \\
&\quad \left. - \sum_{l=0}^{k-2} \left[ \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{k-l-1}} \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{l+1}} + \frac{\partial^2\varphi_s(\mathbf{q})}{\partial q_{k-l-1}\partial q_{l+1}} \right] \right\} \\
&\quad + \sum_{k,l=1}^{\infty} klq_kq_l \left[ \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{k+l+1}} - \frac{\partial\varphi_s(\mathbf{q})}{\partial q_{k+l}} \right]. \quad (2.20)
\end{aligned}$$

Since, for  $s = 0$ , Eq. (2.10) gives  $T_k = N$ , Eqs. (2.12) and (2.13) give the initial condition

$$\varphi_{s=0}(\mathbf{q}) = -N \sum_{k=1}^{\infty} q_k. \quad (2.21)$$

Notice also, from (2.12) and (2.13), the relation

$$\varphi_s(\mathbf{q} = \mathbf{0}) = 0. \quad (2.22)$$

Our purpose, in the next section, is the analysis of the CGF, whose evolution is described by Eqs. (2.20) and (2.21).

### III. EXPANSION OF THE CGF IN INVERSE POWERS OF $N$

Assuming that the number of channels  $N \gg 1$  we propose, for the CGF introduced in the previous section, the expansion

$$\varphi_s(\mathbf{q}) = \sum_{m=-\infty}^M \varphi_s^{(m)}(\mathbf{q}) N^m, \quad (3.1)$$

in decreasing powers of  $N$ . The highest power occurring in (3.1) will be found below to be  $M = 1$ . As an indication, notice that  $\varphi_s \sim N$  gives a contribution  $\sim N^2$  on both sides of the evolution equation (2.20), whereas  $\varphi_s \sim N^2$ , say, would give a contribution  $\sim N^3$  on the left and  $\sim N^4$  on the right-hand side.

The initial condition (2.21) translates, in terms of the functions  $\varphi_s^{(m)}(\mathbf{q})$  of Eq. (3.1), into

$$\varphi_{s=0}^{(m)}(\mathbf{q}) = -\delta_{m1} \sum_{k=1}^{\infty} q_k. \quad (3.2)$$

We assume  $M \geq 1$  and plug the expansion (3.1) into the evolution equation (2.20). We find

$$\begin{aligned}
\sum_{m=-\infty}^M [\partial_s \varphi_s^{(m)}] N^{m+1} &= \sum_{k=1}^{\infty} kq_k \left\{ \sum_{l=0}^{k-1} \left[ \sum_{m',m''=-\infty}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l}} \frac{\partial\varphi_s^{(m'')}}{\partial q_{l+1}} N^{m'+m''} + \sum_{m=-\infty}^M \frac{\partial^2\varphi_s^{(m)}}{\partial q_{k-l}\partial q_{l+1}} N^m \right] \right. \\
&\quad \left. - \sum_{l=0}^{k-2} \left[ \sum_{m',m''=-\infty}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l-1}} \frac{\partial\varphi_s^{(m'')}}{\partial q_{l+1}} N^{m'+m''} + \sum_{m=-\infty}^M \frac{\partial^2\varphi_s^{(m)}}{\partial q_{k-l-1}\partial q_{l+1}} N^m \right] \right\} \\
&\quad + \sum_{k,l=1}^{\infty} klq_kq_l \sum_{m=-\infty}^M \left[ \frac{\partial\varphi_s^{(m)}}{\partial q_{k+l+1}} - \frac{\partial\varphi_s^{(m)}}{\partial q_{k+l}} \right] N^m. \quad (3.3)
\end{aligned}$$

On the right-hand side we change the summation indices  $m', m''$  to  $m = m' + m'' - 1$  and  $m'$  (that run, respectively, from  $-\infty$  to  $2M-1$  and from  $m-M+1$  to  $M$ ) and the summation index  $m$  to  $m+1$  (the new  $m$  now running from  $-\infty$  to  $M-1$ ). Equating the coefficients of powers of  $N$  we find three groups of equations: For  $m = M+1, \dots, 2M-1$ :

$$\sum_{k=1}^{\infty} kq_k \left[ \sum_{l=0}^{k-1} \sum_{m'=m-M+1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l}} \frac{\partial\varphi_s^{(m-m'+1)}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} \sum_{m'=m-M+1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l-1}} \frac{\partial\varphi_s^{(m-m'+1)}}{\partial q_{l+1}} \right] = 0; \quad (3.4a)$$

for  $m = M$ :

$$\partial_s \varphi_s^{(M)} = \sum_{k=1}^{\infty} kq_k \left[ \sum_{l=0}^{k-1} \sum_{m'=1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l}} \frac{\partial\varphi_s^{(M+1-m')}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} \sum_{m'=1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l-1}} \frac{\partial\varphi_s^{(M+1-m')}}{\partial q_{l+1}} \right]; \quad (3.4b)$$

for  $m \leq M-1$ :

$$\begin{aligned}
\partial_s \varphi_s^{(m)} &= \sum_{k=1}^{\infty} kq_k \left\{ \sum_{l=0}^{k-1} \left[ \sum_{m'=m-M+1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l}} \frac{\partial\varphi_s^{(m-m'+1)}}{\partial q_{l+1}} + \frac{\partial^2\varphi_s^{(m+1)}}{\partial q_{k-l}\partial q_{l+1}} \right] \right. \\
&\quad \left. - \sum_{l=0}^{k-2} \left[ \sum_{m'=m-M+1}^M \frac{\partial\varphi_s^{(m')}}{\partial q_{k-l-1}} \frac{\partial\varphi_s^{(m-m'+1)}}{\partial q_{l+1}} + \frac{\partial^2\varphi_s^{(m+1)}}{\partial q_{k-l-1}\partial q_{l+1}} \right] \right\} + \sum_{k,l=1}^{\infty} klq_kq_l \left[ \frac{\partial\varphi_s^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial\varphi_s^{(m+1)}}{\partial q_{k+l}} \right]. \quad (3.4c)
\end{aligned}$$

We consider two examples, whose generalization will be quite clear. First, take  $M = 2$ . Equation (3.4a) gives, for  $m = 3$ :

$$\sum_{k=1}^{\infty} k q_k \left[ \sum_{l=0}^{k-1} \frac{\partial \varphi_s^{(2)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(2)}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} \frac{\partial \varphi_s^{(2)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(2)}}{\partial q_{l+1}} \right] = 0 \quad (3.5)$$

while (3.4b) gives, for  $m = 2$ :

$$\partial_s \varphi_s^{(2)} = \sum_{k=1}^{\infty} k q_k \left[ \sum_{l=0}^{k-1} \sum_{m'=1}^2 \frac{\partial \varphi_s^{(m')}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(3-m')}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} \sum_{m'=1}^2 \frac{\partial \varphi_s^{(m')}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(3-m')}}{\partial q_{l+1}} \right] \quad (3.6)$$

The right-hand side of (3.6) contain  $\varphi_s^{(1)} \varphi_s^{(2)}$ . Suppose we knew  $\varphi_s^{(1)}$ ; then (3.6) could be regarded as a linear equation, *homogeneous* in  $\varphi_s^{(2)}$ , first order in the evolution parameter  $s$ , with zero initial condition [from (3.2)]

$$\varphi_{s=0}^{(2)}(\mathbf{q}) = 0 \quad (3.7)$$

The solution of (3.6) is then

$$\varphi_s^{(2)}(\mathbf{q}) \equiv 0 \quad \forall s \quad (3.8)$$

which also satisfies (3.5) identically.

As a second example, consider  $M = 3$ . Equation (3.4b) for  $m = 3$  is the evolution equation for  $\varphi_s^{(3)}$  and contains, on the right-hand side,  $\varphi_s^{(3)} \varphi_s^{(1)}$  and  $\varphi_s^{(2)} \varphi_s^{(2)}$ ; Eq. (3.4c) for  $m = 2$  is the evolution equation for  $\varphi_s^{(2)}$  and contains, on the right-hand side,  $\varphi_s^{(3)} \varphi_s^{(0)}$ ,  $\varphi_s^{(2)} \varphi_s^{(1)}$ , and  $\varphi_s^{(3)}$ . Suppose we knew  $\varphi_s^{(1)}$ ,  $\varphi_s^{(0)}$ ; the two evolution equations just described are nonlinear, *homogeneous* equations in  $\varphi_s^{(2)}$ ,  $\varphi_s^{(3)}$ , first order in the evolution parameter  $s$ , with zero initial conditions [from (3.2)]

$$\varphi_{s=0}^{(2)}(\mathbf{q}) = \varphi_{s=0}^{(3)}(\mathbf{q}) = 0 \quad (3.9)$$

We then have the solution

$$\varphi_s^{(2)}(\mathbf{q}) = \varphi_s^{(3)}(\mathbf{q}) \equiv 0 \quad \forall s \quad (3.10)$$

Equation (3.4a) for  $m = 4, 5$  contain only  $\varphi_s^{(3)} \varphi_s^{(2)}$  and  $\varphi_s^{(3)} \varphi_s^{(3)}$ , respectively, and are thus identically satisfied.

The above scheme can obviously be generalized, with the conclusion that the maximum power occurring in the expansion (3.1) is  $M = 1$ . In this case (3.4a) does not contribute, while (3.4b) and (3.4c) give Eqs. (3.11a) and (3.15a) below, respectively.

In previous publications,<sup>7,8</sup> diffusion equations in the

variable  $s$  were solved by using the initial condition for the quantity in question and its derivatives at  $s = 0$  (obtained by successive differentiations of the diffusion equation) to obtain a series expansion that could be summed within the radius of convergence and analytically continued outside. A similar procedure will be used below to find, for  $s > 0$ , the structure of the solution  $\varphi_s^{(m)}(\mathbf{q})$  in the variables  $\mathbf{q}$ .

**A. Evolution equation for  $\varphi_s^{(1)}(\mathbf{q})$ :  
Structure of the solution  
in the variables  $q_k$ 's**

From (3.4b) with  $M = 1$  we have

$$\partial_s \varphi_s^{(1)}(\mathbf{q}) = \sum_{k=1}^{\infty} k q_k \left[ \sum_{l=0}^{k-1} \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(1)}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(1)}}{\partial q_{l+1}} \right] \quad (3.11a)$$

with the initial condition [see Eq. (3.2)]

$$\varphi_{s=0}^{(1)}(\mathbf{q}) = - \sum_{k=1}^{\infty} q_k \quad (3.11b)$$

Evaluating (3.11a) at  $s = 0$ , we find

$$\dot{\varphi}_{s=0}^{(1)}(\mathbf{q}) = \sum_{k=1}^{\infty} k q_k \quad (3.12)$$

Successive derivatives with respect to  $s$  of (3.11a), evaluated at  $s = 0$ , give again expressions linear in the  $q_k$ 's. The resulting  $\varphi_s^{(1)}(\mathbf{q})$  for  $s > 0$  is thus *linear* in the  $q_k$ 's, i.e.,

$$\varphi_s^{(1)}(\mathbf{q}) = \sum_{k=1}^{\infty} A_k^{(1)}(s) q_k \quad (3.13)$$

Notice that (3.11b) and (3.12) allow writing  $\varphi_s^{(1)}(\mathbf{q})$  for  $s \ll 1$  as

$$\varphi_s^{(1)}(\mathbf{q}) = \sum_{k=1}^{\infty} (-1 + ks + \dots) q_k \quad (3.14)$$

**B. Evolution equation for  $\varphi_s^{(m)}$ ,  $m \leq 0$ :  
Structure of the solution  
in the variables  $q_k$ 's**

From (3.4c) with  $M = 1$  we have

$$\begin{aligned}
\partial_s \varphi_s^{(m)} = & \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-1} \left[ \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots \right. \right. \\
& \left. \left. + \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(m)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(1)}}{\partial q_{l+1}} \right) + \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\
& - \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-2} \left[ \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots \right. \right. \\
& \left. \left. + \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(m)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(1)}}{\partial q_{l+1}} \right) + \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l}} \right], \tag{3.15a}
\end{aligned}$$

with the initial condition

$$\varphi_{s=0}^{(m)}(\mathbf{q}) = 0, \quad m \leq 0 \tag{3.15b}$$

Consider the  $2 - m$  terms contained in the round parentheses inside the first square bracket in (3.15a). The contributions to the  $l$  summation of the first and last terms are identical, and so are those of the second term and the one before the last, etc., as one can show by changing the  $l$ -summation index to  $l' = k - l - 1$ . A similar conclusion applies to the round parentheses inside the second square bracket, as one can see with the change of variable  $l \rightarrow l' = k - l - 2$ . One can thus write, for  $\varphi_s^{(m)}(\mathbf{q})$ ,  $m \leq 0$ , the evolution equation

$$\begin{aligned}
\partial_s \varphi_s^{(m)} = & \sum_{k=1}^{\infty} k \sum_{l=0}^{k-1} \left[ 2q_k \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots + \frac{\partial \varphi_s^{(\frac{m+2}{2})}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(\frac{m}{2})}}{\partial q_{l+1}} \right) + q_k \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\
& - \sum_{k=1}^{\infty} k \sum_{l=0}^{k-2} \left[ 2q_k \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots + \frac{\partial \varphi_s^{(\frac{m+2}{2})}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(\frac{m}{2})}}{\partial q_{l+1}} \right) \right. \\
& \left. + q_k \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l}} \right], \tag{3.16a}
\end{aligned}$$

for  $m$  even, and

$$\begin{aligned}
\partial_s \varphi_s^{(m)} = & \sum_{k=1}^{\infty} k \sum_{l=0}^{k-1} \left[ 2q_k \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots + \frac{\partial \varphi_s^{(\frac{m+3}{2})}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(\frac{m-1}{2})}}{\partial q_{l+1}} \right) \right. \\
& \left. + q_k \frac{\partial \varphi_s^{(\frac{m+1}{2})}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(\frac{m+1}{2})}}{\partial q_{l+1}} + q_k \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\
& - \sum_{k=1}^{\infty} k \sum_{l=0}^{k-2} \left[ 2q_k \left( \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{l+1}} + \dots + \frac{\partial \varphi_s^{(\frac{m+3}{2})}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(\frac{m-1}{2})}}{\partial q_{l+1}} \right) \right. \\
& \left. + q_k \frac{\partial \varphi_s^{(\frac{m+1}{2})}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(\frac{m+1}{2})}}{\partial q_{l+1}} + q_k \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+l}} \right], \tag{3.16b}
\end{aligned}$$

for  $m$  odd.

It is illustrative to write (3.16) for the specific cases  $m = 0, -1$ , explicitly. We find

$$\begin{aligned}
\partial_s \varphi_s^{(0)} = & \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-1} \left[ 2 \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial^2 \varphi_s^{(1)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\
& - \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-2} \left[ 2 \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial^2 \varphi_s^{(1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial \varphi_s^{(1)}}{\partial q_{k+l+1}} - \frac{\partial \varphi_s^{(1)}}{\partial q_{k+l}} \right], \tag{3.17}
\end{aligned}$$

$$\begin{aligned} \partial_s \varphi_s^{(-1)} = & \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-1} \left[ 2 \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(-1)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial^2 \varphi_s^{(0)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\ & - \sum_{k=1}^{\infty} k q_k \sum_{l=0}^{k-2} \left[ 2 \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(-1)}}{\partial q_{l+1}} + \frac{\partial \varphi_s^{(0)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(0)}}{\partial q_{l+1}} + \frac{\partial^2 \varphi_s^{(0)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] \\ & + \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial \varphi_s^{(0)}}{\partial q_{k+l+1}} - \frac{\partial \varphi_s^{(0)}}{\partial q_{k+l}} \right] \end{aligned} \tag{3.18}$$

Consider (3.17) first. Since  $\varphi_s^{(1)}$  is linear in  $\mathbf{q}$ , Eq. (3.13),  $\partial \varphi_s^{(1)} / \partial q_k = A_k^{(1)}(s)$ ,  $\partial^2 \varphi_s^{(1)} / \partial q_k \partial q_l = 0$ . From the initial condition (3.15b), i.e.,  $\varphi_{s=0}^{(0)}(\mathbf{q}) = 0$ , we have  $\partial \varphi_{s=0}^{(0)}(\mathbf{q}) / \partial q_k = 0$ ; Eq. (3.17) evaluated at  $s = 0$  then implies that  $\dot{\varphi}_{s=0}^{(0)}(\mathbf{q})$  (the dot indicating differentiation with respect to  $s$ ) is quadratic in  $\mathbf{q}$ ; differentiating (3.17) with respect to  $s$  we find that  $\ddot{\varphi}_{s=0}^{(0)}(\mathbf{q})$  is also quadratic in  $\mathbf{q}$ , and similarly for higher derivatives at  $s = 0$  [an argument by induction is given right after Eq. (3.20)]. We thus reach the conclusion that  $\varphi_s^{(0)}(\mathbf{q})$  itself is quadratic in  $\mathbf{q}$ , i.e.,

$$\varphi_s^{(0)}(\mathbf{q}) = \sum_{k,l=1}^{\infty} A_{kl}^{(0)}(s) q_k q_l \tag{3.19}$$

We now turn to (3.18), using again the linearity of  $\varphi_s^{(1)}$  in  $\mathbf{q}$ , using Eq. (3.19) for  $\varphi_s^{(0)}$  and the initial condition (3.15b), i.e.,  $\varphi_{s=0}^{(-1)}(\mathbf{q}) = 0$ . Again, one can convince oneself (a proof by induction is given below) that (3.18) and its successive derivatives with respect to  $s$  evaluated at  $s = 0$  only contain linear and cubic terms in the  $q_k$ 's; thus,

$$\varphi_s^{(-1)} = \sum_{k=1}^{\infty} A_k^{(-1)}(s) q_k + \sum_{k,l,m=1}^{\infty} A_{klm}^{(-1)}(s) q_k q_l q_m \tag{3.20}$$

The  $q$  structure of  $\varphi_s^{(m)}(\mathbf{q})$  for arbitrary  $m \leq 0$  can now be found by induction as follows. Suppose we have proved that

$$\varphi_s^{(j)}(\mathbf{q}) = \text{odd (even) polynomial in } \mathbf{q} \text{ of degree } 2 - j \text{ for } j \text{ odd (even)}, \tag{3.21}$$

for  $j = 1, 0, -1, \dots, m + 1$ . Odd (even) polynomial means a polynomial containing only odd (even, nonzero) powers of the  $q_k$ 's. We then show below that Eqs. (3.16) imply (3.21) for  $\varphi_s^{(m)}(\mathbf{q})$ . We first collect, on the left-hand side of Eqs. (3.16a,b), the terms that contain the  $\varphi_s^{(m)}(\mathbf{q})$  we are looking for and use Eq. (3.13) to obtain

$$\begin{aligned} \partial_s \varphi_s^{(m)} - 2 \sum_{k=1}^{\infty} k \left[ \sum_{l=0}^{k-1} q_k \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} - \sum_{l=0}^{k-2} q_k \frac{\partial \varphi_s^{(1)}}{\partial q_{k-l-1}} \frac{\partial \varphi_s^{(m)}}{\partial q_{l+1}} \right] \\ = \partial_s \varphi_s^{(m)} - 2 \sum_{k=1}^{\infty} k \left[ \sum_{l=0}^{k-1} A_{k-l}^{(1)}(s) \left( q_k \frac{\partial}{\partial q_{l+1}} \right) \varphi_s^{(m)} - \sum_{l=0}^{k-2} A_{k-l-1}^{(1)}(s) \left( q_k \frac{\partial}{\partial q_{l+1}} \right) \varphi_s^{(m)} \right] \\ = \psi_s^{(m)}(\mathbf{q}) \end{aligned} \tag{3.22}$$

Here  $\psi_s^{(m)}(\mathbf{q})$  involves all the other terms in Eqs. (3.16a,b) containing only  $\varphi_s^{(0)}(\mathbf{q}), \dots, \varphi_s^{(m+1)}(\mathbf{q})$ ;  $\psi_s^{(m)}(\mathbf{q})$  can thus be considered as an *inhomogeneous source term*. From our assumption, (3.21), one can see that the source  $\psi_s^{(m)}(\mathbf{q})$  is a polynomial of degree  $2 - m$  in  $\mathbf{q}$  containing powers of the  $q_k$ 's of the same parity as  $m$  (with no constant term) and thus fulfills Eq. (3.21) for  $j = m$ . Finally, we need to prove (3.21) for  $\varphi_s^{(m)}(\mathbf{q})$ , or, equivalently, for all its derivatives with respect to  $s$  evaluated at  $s = 0$ : we do this by constructing still another induction argument. The  $p$ th derivative of Eq. (3.22) at  $s = 0$ , i.e.,

$$\begin{aligned} \left[ \frac{\partial^{p+1} \varphi_s^{(m)}(\mathbf{q})}{\partial s^{p+1}} \right]_{s=0} - 2 \sum_{k=1}^{\infty} k \left[ \sum_{l=0}^{k-1} q_k \frac{\partial}{\partial q_{l+1}} \sum_{r=0}^p \binom{p}{r} \left( \frac{d^{p-r} A_{k-l}^{(1)}(s)}{ds^{p-r}} \right) \left( \frac{\partial^r \varphi_s^{(m)}}{\partial s^r} \right) \right]_{s=0} \\ - \sum_{l=0}^{k-2} q_k \frac{\partial}{\partial q_{l+1}} \sum_{r=0}^p \binom{p}{r} \left( \frac{d^{p-r} A_{k-l-1}^{(1)}(s)}{ds^{p-r}} \right) \left( \frac{\partial^r \varphi_s^{(m)}}{\partial s^r} \right) \Big|_{s=0} = \left[ \frac{\partial^p \psi_s^{(m)}(\mathbf{q})}{\partial s^p} \right]_{s=0} \end{aligned} \tag{3.23}$$

satisfies (3.21) for  $j = m$ , since  $\psi_s^{(m)}$  does. Using the initial condition  $\varphi_{s=0}^{(m)}(\mathbf{q}) = 0$ ,  $m \leq 0$ , Eq. (3.23) with  $p = 0$  shows that  $\varphi_{s=0}^{(m)}(\mathbf{q})$  satisfies (3.21) for  $j = m$ . Supposing that the first  $p$  derivatives of  $\varphi_s^{(m)}(\mathbf{q})$  with respect to  $s$  at  $s = 0$  satisfy Eq. (3.21) for  $j = m$ , Eq. (3.23) shows that the derivative of order  $p + 1$  satisfies it too. This completes the proof. *The statement that, in full generality,  $\varphi_s^{(m)}(\mathbf{q})$  satisfies Eq. (3.21) for  $j = m$  is the first important result of the present paper.*

We end this section by indicating the relevance of the above considerations to the structure of the various cumulants. A cumulant  $\kappa_{k_1 \dots k_r}$  of order  $l = \sum_{i=1}^r k_i$  of the variables  $T_1, \dots, T_r$  can be found from the coefficient of  $q_1^{k_1} \dots q_r^{k_r}$  in the CGF  $\varphi_s(\mathbf{q})$ . From (3.13), (3.19), (3.20), and the above generalization we thus see that a first-order cumulant contains contributions from  $\varphi_s^{(1)}N$ ,  $\varphi_s^{(-1)}N^{-1}$ , . . . , a second-order one involves  $\varphi_s^{(0)}N^0$ ,  $\varphi_s^{(-2)}N^{-2}$ , . . . , an  $l$ th-order one  $\varphi_s^{(2-l)}N^{2-l}$ ,  $\varphi_s^{(-l)}N^{-l}$ , . . . . In particular, the cumulant of arbitrary order  $l$  of the conductance  $T_1$  behaves as

$$\kappa_l(s) = (-1)^{l!} \left[ \underbrace{A_{1 \dots 1}^{(2-l)}(s)}_{l!} N^{2-l} + \underbrace{A_{1 \dots 1}^{(-l)}(s)}_{l!} N^{-l} + \dots \right]. \quad (3.24)$$

#### IV. $s$ DEPENDENCE OF THE CGF

In this section we shall be concerned with the  $s$  dependence of the CGF components  $\varphi_s^{(m)}(\mathbf{q})$  for  $s \gg 1$ , i.e., in the metallic regime.

##### A. The case $m = 1$

Introducing (3.13) into (3.11a) we find, for the quantities  $A_k^{(1)}(s)$ , the evolution equations

$$\dot{A}_k^{(1)}(s) = k \left[ \sum_{l=0}^{k-1} A_{k-l}^{(1)}(s) A_{l+1}^{(1)}(s) - \sum_{l=0}^{k-2} A_{k-l-1}^{(1)}(s) A_{l+1}^{(1)}(s) \right]. \quad (4.1)$$

Before proceeding, we connect Eq. (4.1) with the results of earlier analysis. In Eq. (B8) of Ref. 8, the evolution equation for  $\langle T_k \rangle$  was found to be

$$N \partial_s \langle T_k \rangle_s = \left\langle k(k-1)T_k - k^2 T_{k+1} + k \left[ \sum_{l=0}^{k-2} (T_{l+1} T_{k-l-1} - T_k) - \sum_{l=0}^{k-1} (T_{l+1} T_{k-l} - T_{k+1}) \right] \right\rangle_s. \quad (4.2)$$

The dominant contribution for  $N \gg 1$  in this equation is

$$N \partial_s \langle T_k \rangle_s^{(N)} = -k \left[ \sum_{l=0}^{k-1} \langle T_{k-l} \rangle_s^{(N)} \langle T_{l+1} \rangle_s^{(N)} - \sum_{l=0}^{k-2} \langle T_{k-l-1} \rangle_s^{(N)} \langle T_{l+1} \rangle_s^{(N)} \right], \quad (4.3)$$

where  $\langle T_k \rangle_s^{(N)}$  designates the term of order  $N$  in an  $N$ -power expansion of  $\langle T_k \rangle$ . Comparing with (4.1) we thus identify

$$A_k^{(1)}(s) = -\frac{\langle T_k \rangle_s^{(N)}}{N} = -\lim_{N \rightarrow \infty} \frac{\langle T_k \rangle_s}{N}, \quad (4.4)$$

as expected.

In particular, from Eq. (5.14) of Ref. 8, we find

$$A_1^{(1)}(s) = -\frac{1}{1+s}, \quad (4.5)$$

from Eq. (C24) of Ref. 8

$$A_2^{(1)}(s) = -\frac{2s^3 + 6s^2 + 6s + 3}{3(1+s)^4}, \quad (4.6)$$

and from Eqs. (5.11) and (C20) of Ref. 8

$$A_3^{(1)}(s) = \frac{8s^6 + 48s^5 + 120s^4 + 165s^3 + 135s^2 + 60s + 15}{15(1+s)^7}. \quad (4.7)$$

To first order in  $s$ , Eqs. (4.5)–(4.7) agree with the relation (3.14). It seems natural to assume that if one should extend the analysis of Refs. 7 and 8 beyond the quantities of Eqs. (4.5)–(4.7), one would generally encounter ratios of polynomials in  $s$ . The asymptotic behavior for  $s \gg 1$  of the various  $s$ -dependent quantities will thus be sought, for  $m = 1, 0, \dots$ , as a power in  $s$ . For  $A_k^{(1)}(s)$ , we thus propose, for  $s \gg 1$ ,

$$A_k^{(1)}(s) \sim a_k^{(1)} s^{r_1}. \quad (4.8)$$

Introducing (4.8) in the evolution equation (4.1), we find

$$r_1 a_k^{(1)} s^{r_1-1} = k s^{2r_1} \left[ \sum_{l=0}^{k-1} a_{k-l}^{(1)} a_{l+1}^{(1)} - \sum_{l=0}^{k-2} a_{k-l-1}^{(1)} a_{l+1}^{(1)} \right]. \quad (4.9)$$

Thus,

$$r_1 = -1 \quad (4.10)$$

and

$$A_k^{(1)}(s) \sim \frac{a_k^{(1)}}{s}. \quad (4.11)$$

For the  $a_k^{(1)}$ 's, we thus have the relations

$$a_k^{(1)} = k \left[ \sum_{l=0}^{k-2} a_{k-l-1}^{(1)} a_{l+1}^{(1)} - \sum_{l=0}^{k-1} a_{k-l}^{(1)} a_{l+1}^{(1)} \right], \quad (4.12)$$

which can be solved iteratively. For  $k = 1$  we have

$$a_1^{(1)} = - \left[ a_1^{(1)} \right]^2, \quad (4.13)$$

with the nonzero solution

$$a_1^{(1)} = -1, \quad (4.14)$$

consistent with (4.5). Next, we obtain

$$a_2^{(1)} = \frac{2 \left[ a_1^{(1)} \right]^2}{1 + 4a_1^{(1)}} = -\frac{2}{3} \quad (4.15)$$

and

$$a_3^{(1)} = \frac{6a_1^{(1)} a_2^{(1)} - 3 \left[ a_2^{(1)} \right]^2}{1 + 6a_1^{(1)}} = -\frac{8}{15}, \quad (4.16)$$

consistent with (4.6) and (4.7), respectively. We further obtain

$$a_4^{(1)} = -\frac{16}{35}, \quad a_5^{(1)} = -\frac{128}{315}. \quad (4.17)$$

Results (4.14)–(4.17) are consistent with Eq. (8) of Ref. 12.

**B. The case  $m = 0$**

Introducing (3.13) and (3.19) in (3.17) we find, for  $A_{kn}^{(0)}(s)$ , the evolution equations

$$\begin{aligned} \dot{A}_{kn}^{(0)}(s) = & 2k \sum_{l=0}^{k-1} A_{k-l}^{(1)}(s) A_{l+1,n}^{(0)}(s) + 2n \sum_{l=0}^{n-1} A_{n-l}^{(1)}(s) A_{l+1,k}^{(0)}(s) - 2k \sum_{l=0}^{k-2} A_{k-l-1}^{(1)}(s) A_{l+1,n}^{(0)}(s) \\ & - 2n \sum_{l=0}^{n-2} A_{n-l-1}^{(1)}(s) A_{l+1,k}^{(0)}(s) + kn \left[ A_{k+n+1}^{(1)}(s) - A_{k+n}^{(1)}(s) \right]. \end{aligned} \quad (4.18)$$

Assuming, as above,

$$A_{kn}^{(0)}(s) \sim a_{kn}^{(0)} s^{r_0} \quad (4.19)$$

for  $s \gg 1$ , we obtain  $r_0 s^{r_0-1}$  on the left-hand side of (4.18) and, on the right-hand side,  $s^{r_0-1}$  and  $s^{-1}$ . Thus,

$$r_0 = 0 \quad (4.20)$$

and

$$A_{kn}^{(0)}(s) \sim a_{kn}^{(0)}. \quad (4.21)$$

For the  $a_{kn}^{(0)}$ 's we find the recursion relations

$$\begin{aligned} & 2k \sum_{l=0}^{k-1} a_{k-l}^{(1)} a_{l+1,n}^{(0)} + 2n \sum_{l=0}^{n-1} a_{n-l}^{(1)} a_{l+1,k}^{(0)} \\ & - 2k \sum_{l=0}^{k-2} a_{k-l-1}^{(1)} a_{l+1,n}^{(0)} - 2n \sum_{l=0}^{n-2} a_{n-l-1}^{(1)} a_{l+1,k}^{(0)} \\ & + kn \left[ a_{k+n+1}^{(1)} - a_{k+n}^{(1)} \right] = 0, \end{aligned} \quad (4.22)$$

which can be solved iteratively, once the  $a_k^{(1)}$  have been found in the previous step. For example, for  $k = n = 1$  we obtain

$$a_{11}^{(0)} = \frac{a_2^{(1)} - a_3^{(1)}}{4a_1^{(1)}} = \frac{1}{30}, \quad (4.23)$$

where we have used (4.14)–(4.16). Thus,  $\text{var}T$ , to order  $N^0$ , is

$$(\text{var}T)^{(N^0)} = 2! A_{11}^{(0)}(s) = \frac{1}{15}, \quad (4.24)$$

a well known result. Similarly, one finds

$$a_{12}^{(0)} = \frac{31}{945}, \quad a_{22}^{(0)} = \frac{103}{2835}, \quad a_{13}^{(0)} = \frac{289}{9450} \quad (4.25)$$

so that

$$\begin{aligned} [\text{cov}(T_1, T_2)]^{(N^0)} &= \frac{62}{945}, \\ (\text{var}T_2)^{(N^0)} &= \frac{206}{2835}, \\ [\text{cov}(T_1, T_3)]^{(N^0)} &= \frac{289}{4725}. \end{aligned} \quad (4.26)$$

Results (4.23)–(4.26) are consistent with those found after Eq. (13) of Ref. 12.

**C. The case  $m = -1$**

We introduce (3.13), (3.19), and (3.20) in (3.18). Now the appropriate dependence on  $s$ , for  $s \gg 1$ , is

$$A_k^{(-1)}(s) \sim a_k^{(-1)} s, \quad (4.27a)$$

$$A_{klm}^{(-1)}(s) \sim a_{klm}^{(-1)} s. \quad (4.27b)$$



For  $a_k^{(-1)}$  we find the recursion relation

$$a_{111}^{(-1)} = 2 \frac{2 [a_{11}^{(0)}]^2 + a_{13}^{(0)} - a_{12}^{(0)}}{1 - 6a_1^{(0)}} \quad (4.30)$$

$$a_k^{(-1)} = 2k \sum_{l=0}^{k-1} [a_{k-l}^{(1)} a_{l+1}^{(-1)} + a_{k-l, l+1}^{(0)}] - 2k \sum_{l=0}^{k-2} [a_{k-l-1}^{(1)} a_{l+1}^{(-1)} + a_{k-l-1, l+1}^{(0)}] \quad (4.28)$$

Using (4.14), (4.23), and (4.25), we then find

$$a_{111}^{(-1)} = 0 \quad (4.31)$$

in agreement with Eq. (2c) (for  $\beta = 2$ ) of Ref. 12.

It is clear that the iteration procedure discussed in Secs. IV A-IV C can be used to find, in a systematic way, the numerical value of any of the coefficients  $a_k^{(1)}$ ,  $a_{kn}^{(0)}$ ,  $a_k^{(-1)}$ , and  $a_{k_1 k_2 k_3}^{(-1)}$ .

One finds, for instance

$$a_1^{(-1)} = \frac{1}{45} \quad (4.29)$$

in agreement with Eq. (5.31) of Ref. 8. We do not quote the recursion relation for  $a_{klm}^{(-1)}$ ; we just mention that one obtains, from it

**D. The case of arbitrary  $m$**

We now show that the asymptotic ( $s \gg 1$ ) relations (4.11), (4.21), and (4.27) can be generalized to arbitrary  $m$ . Inspired by the above analysis we propose

$$\varphi_s^{(m)}(\mathbf{q}) \sim f^{(m)}(\mathbf{q}) s^{r_m} \quad (4.32)$$

Substitution in the evolution equations (3.16) for  $m \leq 0$  gives

$$\begin{aligned} r_m f^{(m)}(\mathbf{q}) s^{r_m-1} = & \sum_{k=1}^{\infty} k \sum_{l=0}^{k-1} \left[ 2q_k \left( s^{r_1+r_m} \frac{\partial f^{(1)}}{\partial q_{k-l}} \frac{\partial f^{(m)}}{\partial q_{l+1}} + s^{r_0+r_{m+1}} \frac{\partial f^{(0)}}{\partial q_{k-l}} \frac{\partial f^{(m+1)}}{\partial q_{l+1}} + \dots \right. \right. \\ & \left. \left. + s^{r_{(m+2)/2+r_m/2}} \frac{\partial f^{(\frac{m+2}{2})}}{\partial q_{k-l}} \frac{\partial f^{(\frac{m}{2})}}{\partial q_{l+1}} \right) + q_k s^{r_{m+1}} \frac{\partial^2 f^{(m+1)}}{\partial q_{k-l} \partial q_{l+1}} \right] \\ & - \sum_{k=1}^{\infty} k \sum_{l=0}^{k-2} \left[ 2q_k \left( s^{r_1+r_m} \frac{\partial f^{(1)}}{\partial q_{k-l-1}} \frac{\partial f^{(m)}}{\partial q_{l+1}} + s^{r_0+r_{m+1}} \frac{\partial f^{(0)}}{\partial q_{k-l-1}} \frac{\partial f^{(m+1)}}{\partial q_{l+1}} + \dots \right. \right. \\ & \left. \left. + s^{r_{(m+2)/2+r_m/2}} \frac{\partial f^{(\frac{m+2}{2})}}{\partial q_{k-l-1}} \frac{\partial f^{(\frac{m}{2})}}{\partial q_{l+1}} \right) + q_k s^{r_{m+1}} \frac{\partial^2 f^{(m+1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] \\ & + s^{r_{m+1}} \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial f^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial f^{(m+1)}}{\partial q_{k+l}} \right] \quad (4.33a) \end{aligned}$$

for  $m$  even, and

$$\begin{aligned} r_m f^{(m)}(\mathbf{q}) s^{r_m-1} = & \sum_{k=1}^{\infty} k \sum_{l=0}^{k-1} \left[ 2q_k \left( s^{r_1+r_m} \frac{\partial f^{(1)}}{\partial q_{k-l}} \frac{\partial f^{(m)}}{\partial q_{l+1}} + s^{r_0+r_{m+1}} \frac{\partial f^{(0)}}{\partial q_{k-l}} \frac{\partial f^{(m+1)}}{\partial q_{l+1}} + \dots \right. \right. \\ & \left. \left. + s^{r_{(m+3)/2+r_{(m-1)/2}} \frac{\partial f^{(\frac{m+3}{2})}}{\partial q_{k-l}} \frac{\partial f^{(\frac{m-1}{2})}}{\partial q_{l+1}} \right) + q_k s^{2r_{(m+1)/2}} \frac{\partial f^{(\frac{m+1}{2})}}{\partial q_{k-l}} \frac{\partial f^{(\frac{m+1}{2})}}{\partial q_{l+1}} \right. \\ & \left. + q_k s^{r_{m+1}} \frac{\partial^2 f^{(m+1)}}{\partial q_{k-l} \partial q_{l+1}} \right] - \sum_{k=1}^{\infty} k \sum_{l=0}^{k-2} \left[ 2q_k \left( s^{r_1+r_m} \frac{\partial f^{(1)}}{\partial q_{k-l-1}} \frac{\partial f^{(m)}}{\partial q_{l+1}} \right. \right. \\ & \left. \left. + s^{r_0+r_{m+1}} \frac{\partial f^{(0)}}{\partial q_{k-l-1}} \frac{\partial f^{(m+1)}}{\partial q_{l+1}} + \dots + s^{r_{(m+3)/2+r_{(m-1)/2}} \frac{\partial f^{(\frac{m+3}{2})}}{\partial q_{k-l-1}} \frac{\partial f^{(\frac{m-1}{2})}}{\partial q_{l+1}} \right) \right. \\ & \left. + q_k s^{2r_{(m+1)/2}} \frac{\partial f^{(\frac{m+1}{2})}}{\partial q_{k-l-1}} \frac{\partial f^{(\frac{m+1}{2})}}{\partial q_{l+1}} + q_k s^{r_{m+1}} \frac{\partial^2 f^{(m+1)}}{\partial q_{k-l-1} \partial q_{l+1}} \right] \\ & + s^{r_{m+1}} \sum_{k,l=1}^{\infty} k l q_k q_l \left[ \frac{\partial f^{(m+1)}}{\partial q_{k+l+1}} - \frac{\partial f^{(m+1)}}{\partial q_{k+l}} \right] \quad (4.33b) \end{aligned}$$

for  $m$  odd.

Suppose we have proved

$$r_n = -n, \quad (4.34)$$

for  $n = 1, 0, \dots, m+1$ . From (4.33) we can then find  $r_m$ . The left-hand side of (4.33) goes as  $s^{r_m-1}$ ; on the right-hand side, the first term inside the first or second square brackets goes as  $s^{r_m-1}$ ; the following ones contain  $r_0, r_{-1}, \dots, r_{\frac{m+2}{2}}, r_{\frac{m}{2}}, r_{\frac{m-2}{2}}, \dots, r_{m+1}$  for  $m$  even and  $r_0, r_{-1}, \dots, r_{\frac{m+3}{2}}, r_{\frac{m+1}{2}}, r_{\frac{m-1}{2}}, \dots, r_{m+1}$  for  $m$  odd, which are all known by assumption; all these terms go as  $s^{-m-1}$  and so does the third square bracket. We thus conclude that

$$r_m = -m \quad (4.35)$$

and thus

$$\varphi_s^{(m)}(\mathbf{q}) \sim f^{(m)}(\mathbf{q})s^{-m} \quad (4.36)$$

for  $s \gg 1$ . Equation (4.36) constitutes the second important result of the present paper. All the  $\mathbf{q}$  dependence of  $\varphi_s^{(m)}(\mathbf{q})$  is in the function  $f^{(m)}(\mathbf{q})$  of (4.36), which satisfies (3.21) for  $j = m$ , i.e.,  $f^{(m)}(\mathbf{q})$  is a polynomial of degree  $2 - m$  in the  $q_k$ 's, containing only odd (even  $\neq 0$ ) powers if  $m$  is odd (even); the coefficients  $a_{k_1 k_2 \dots}^{(m)}$  in this polynomial are related (for  $s \gg 1$ ) with the coefficients  $A_{k_1 k_2 \dots}^{(m)}(s)$  by

$$A_{k_1 k_2 \dots}^{(m)}(s) \sim a_{k_1 k_2 \dots}^{(m)} s^{-m}, \quad (4.37)$$

which generalizes (4.11), (4.21), and (4.27).

As a consequence of (4.37), the  $l$ th cumulant of the conductance (3.24) behaves, in the metallic regime, as

$$\begin{aligned} \kappa_l(s) \sim (-1)^l l! \left[ \left( \frac{a_{1 \dots 1}^{(2-l)}}{s^{2-l}} + \frac{b_{1 \dots 1}^{(2-l)}}{s^{3-l}} + \dots \right) N^{2-l} \right. \\ \left. + \left( \frac{a_{1 \dots 1}^{(-l)}}{s^{-l}} + \frac{b_{1 \dots 1}^{(-l)}}{s^{1-l}} + \dots \right) N^{-l} + \dots \right], \quad (4.38a) \end{aligned}$$

$$\begin{aligned} = (-1)^l l! \left[ \left( \frac{a_{1 \dots 1}^{(2-l)}}{s^{2-l}} + \frac{b_{1 \dots 1}^{(2-l)}}{s} + \dots \right) g_0^{2-l} \right. \\ \left. + \left( \frac{a_{1 \dots 1}^{(-l)}}{s^{-l}} + \frac{b_{1 \dots 1}^{(-l)}}{s} + \dots \right) g_0^{-l} + \dots \right], \quad (4.38b) \end{aligned}$$

where we have defined

$$g_0 = \frac{N}{s}. \quad (4.39)$$

Now, suppose we take the limit

$$N \rightarrow \infty, \quad s \rightarrow \infty, \quad (4.40)$$

keeping  $g_0$  of (4.39) fixed; from (4.38b) we obtain

$$\kappa_l(s) \sim (-1)^l l! \left[ a_{1 \dots 1}^{(2-l)} g_0^{2-l} + a_{1 \dots 1}^{(-l)} g_0^{-l} + \dots \right], \quad (4.41)$$

which gives  $\kappa_l(s)$  as an expansion in decreasing powers of  $g_0$ , whose leading term coincides with the result (1.1)

obtained from diagrammatic methods. The fact that  $\kappa_l$  contains only even (odd) powers of  $g_0$  for  $l$  even (odd) was found in Ref. 12 for the first few terms in the expansion of  $\kappa_1, \kappa_2, \kappa_3$ , for which we have

$$\kappa_1(s) \sim - \left[ a_1^{(1)} g_0 + a_1^{(-1)} g_0^{-1} + a_1^{(-3)} g_0^{-3} + \dots \right], \quad (4.42a)$$

$$\kappa_2(s) \sim 2! \left[ a_{11}^{(0)} + a_{11}^{(-2)} g_0^{-2} + \dots \right], \quad (4.42b)$$

$$\kappa_3(s) \sim -3! \left[ a_{111}^{(-1)} g_0^{-1} + a_{111}^{(-3)} g_0^{-3} + \dots \right], \quad (4.42c)$$

with  $a_1^{(1)} = -1$ ,  $a_1^{(-1)} = 1/45$ ,  $2!a_{11}^{(0)} = 1/15$ ,  $a_{111}^{(-1)} = 0$ , etc. If  $g_0 \rightarrow \infty$ , the cumulants of order higher than the second vanish and the distribution of the variable

$$x = \frac{g_0 - \kappa_1(s)}{\sqrt{1/15}} \quad (4.43)$$

becomes a zero-centered Gaussian with unit variance (see also Ref. 15). If  $g_0 > 1$  and finite, we have the corrections to the Gaussian cumulants indicated in (4.41) and (4.42).

The structure (4.41) applies, for  $g_0 > 1$ , to cumulants of all orders; thus, in the model for quasi-one-dimensional disordered systems provided by the diffusion equation (2.4), we do not find the behavior of high-order cumulants obtained in Ref. 1 (for a  $d$ -dimensional cube of side  $L$ ) and indicated right after Eq. (1.1).

## V. SUMMARY AND CONCLUSIONS

The behavior of the cumulants of the conductance of a quasi-one-dimensional mesoscopic system is analyzed, as a function of the number of channels  $N$  and the length  $s = L/\ell$ , within the diffusion equation description, which starts out from a random-matrix local approach. Although the study is performed for the case of unitary symmetry ( $\beta = 2$ ), it could, in principle, be generalized to arbitrary  $\beta$ . The analysis of cumulants of arbitrary order is carried out by means of the CGF  $\varphi_s(\mathbf{q})$ , defined in Eq. (2.13), whose evolution, with increasing length  $s$  of the system, is obtained from the diffusion equation and is given by Eq. (2.20). Assuming that the number of channels  $N$  is large, we expand the CGF  $\varphi_s(\mathbf{q})$  in a series in decreasing powers of  $N$  with coefficients  $\varphi_s^{(m)}(\mathbf{q})$ , as in Eq. (3.1). The evolution equation of the  $\varphi_s^{(m)}(\mathbf{q})$ 's is given in Eqs. (3.11) and (3.16). We prove the fundamental result that  $\varphi_s^{(m)}(\mathbf{q})$  is a polynomial in  $\mathbf{q}$  of order  $2 - m$  containing odd (even  $\neq 0$ ) powers, for  $m$  odd (even) [see (3.21)]. Thus, the  $1/N$  expansion of the  $l$ th-order cumulant  $\kappa_l$  of the conductance starts with the term  $N^{2-l}$ , as shown in Eq. (3.24). Since we are interested in the metallic regime, where  $s \gg 1$  (i.e.,  $L \gg \ell$ ), the coefficient of the term  $N^k$  in the  $1/N$  expansion can itself be expanded in decreasing powers of  $s$  [see Eq. (4.38a)]: We prove that  $1/s^k$  occurs in leading order.

In the limit  $N \rightarrow \infty$ ,  $s \rightarrow \infty$ , with  $g_0 = N/s$  fixed,  $\kappa_l$  can be written as a series in powers of  $g_0 > 1$ , as in Eq. (4.41), whose leading term coincides with the result (1.1). As  $g_0 \rightarrow \infty$ , all cumulants of order higher than the second vanish, giving rise to a Gaussian distribution,

while for  $g_0 > 1$  and finite we have corrections to the Gaussian cumulants.

As for the numerical coefficients of the expansion described above, an explicit evaluation was done in the text for some of them, for the cases  $m = 1, 0, -1$ . The generalization to arbitrary  $m$  of the iteration procedure setup in the text, although tedious to be carried out analytically, is sufficiently systematic to be amenable to a computer analysis. First, one would have to generate, from (3.16), the evolution equation for the coefficients  $A^{(m)}$ 's [as (4.18) for  $m = 0$ ], and then, from (4.37), the recursion

relations for the coefficients  $a^{(m)}$  [as (4.22) for  $m = 0$  and (4.28) for  $m = -1$ ]. These could then be solved numerically. The  $l$ th-order cumulant of the conductance would then follow from Eq. (4.41).

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