PHY3323: ELECTROMAGNETISM I

Exam 1, October 13, 2008.

PROBLEM 1

Consider the vector field

$$\vec{A} = r^2 \cos\theta \hat{r} + r^2 \cos\phi \hat{\theta} - r^2 \cos\theta \sin\theta \hat{\phi} \tag{1}$$

1. Compute the line integral of $\vec{A}$ around the path shown in Fig. 1. The points are labeled by their Cartesian coordinates.
2. Compute the flux of of the vector field $\vec{A}$ through the shaded area (1/8 of a spherical shell).
3. Can the vector field $\vec{A}$ represent an electrostatic field? Why?

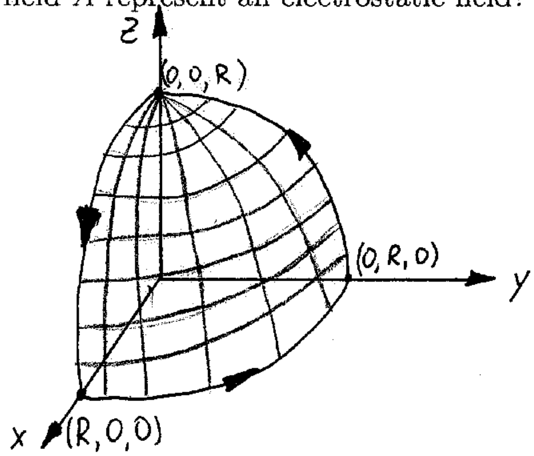

Fig. 1

PROBLEM 2

A non-conducting sphere of radius R caries a charge density

$$\rho = \frac{k}{r} \tag{2}$$

in the region $r \leq R$, where k is a positive constant and and r is the distance from the center of the sphere (see Fig. 2 on the next page). The sphere is surrounded by a thick, concentric metal shell with an inner radius a and an outer radius b. The shell carries no net charge.

1. Find the surface charge density at the inner and the outer surface of the conductor.
2. Find the electric field $\vec{E}$ in all four regions (i) $r < R$; (ii) $R < r < a$, (iii) $a < r < b$, and (iv)$r > b$
3. Find the potential V at the center of the sphere using infinity ($r = \infty$) as a reference point.
4. If the outer shell is grounded, what would be the potential at the center of the sphere using the same reference point as in 3.

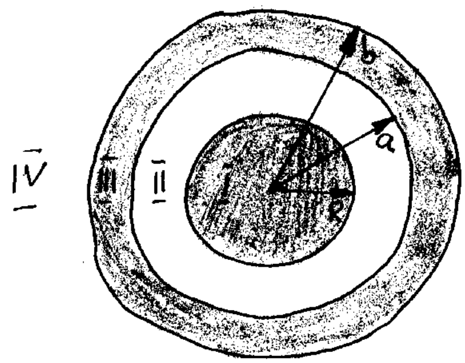

Fig. 2 (Problem 2)

Solution to Problem 1

(a) $\oint \vec{A}\cdot d\vec{l} = \int_{I} \vec{A}\cdot d\vec{l} + \int_{II} \vec{A}\cdot d\vec{l} + \int_{III} \vec{A}\cdot d\vec{l}$

I: $d\vec{l} = r\,d\phi\,\hat{\phi}$ $\qquad r=R$, $\theta=\frac{\pi}{2}$ $\cos\theta=0$

$A_\phi = -R^2\cos\frac{\pi}{2}\sin\phi = 0$

$$I = \int_{\phi=0}^{\pi/2} A_\phi R\,d\phi = 0$$

II: $d\vec{l} = R\,d\theta\,\hat{\theta}$ $\qquad r=R$, $\phi=\frac{\pi}{2}$, $\sin\theta=1$ $\cos\theta=0$

$A_\theta = R^2\cos\frac{\pi}{2} = 0$

$$II = \int_{\theta=0}^{\pi/2} A_\theta R\,d\theta = 0$$

III: $d\vec{l} = R\,d\theta\,\hat{\theta}$ $\qquad r=R$ $\phi=0$, $\sin\theta=0$, $\cos\theta=1$

$A_\theta = R^2\cos\phi = R^2$

$$III = \int_{\theta=0}^{\pi/2} R^2\,R\,d\theta = \frac{\pi R^3}{2}$$

$$\oint \vec{A}\cdot d\vec{l} = 0+0+\frac{\pi}{2}R^3$$

(b) $\Phi = \int \vec{A}\cdot d\vec{a}$ $\qquad d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$A_r = r^2\cos\theta = R^2\cos\theta$

$$\Phi = \int \vec{A}\cdot d\vec{a} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} A_r R^2\sin\theta\,d\theta\,d\phi = R^4\iint \cos\theta\,\sin\theta\,d\theta\,d\phi$$

$$\Phi = R^4 \int_0^{\pi/2}\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \, d\phi = -\frac{\pi}{2} R^4 \int_0^{\pi/2} \cos\theta \, d\cos\theta = -\frac{\pi}{2} R^4 \left. \frac{\cos^2\theta}{2} \right|_0^{\pi/2}$$

$$\boxed{\Phi = \frac{\pi}{4} R^4}$$

(c) No, because $\oint \vec{A} \cdot \vec{dl} \neq 0$

Solution to Problem 2

(a) $\rho = \frac{k}{r}$

$$Q_{sphere} = \int \rho \, d\tau = \iiint \frac{k}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi = 4\pi k \int_0^R r \, dr$$

$$\boxed{Q_{sph} = 2\pi k R^2}$$

$$\sigma_a = -\frac{Q_{sp}}{4\pi a^2} = \boxed{-\frac{kR^2}{2a^2}}$$

$$\boxed{\sigma_b = \frac{kR^2}{2b^2}}$$

(b) $\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$ $\qquad Q_{enc} = 2\pi k r^2$

$$E \, 4\pi r^2 = \frac{2\pi k r^2}{\varepsilon_0} \qquad \boxed{E = \frac{k}{2\varepsilon_0}} = \text{const} \qquad \underline{\underline{r < R}}$$

$R < r < a$ $\qquad E = \frac{Q_{sph}}{4\pi\varepsilon_0 r^2} = \frac{2\pi k R^2}{4\pi\varepsilon_0 r^2} = \frac{k}{2\varepsilon_0}\frac{R^2}{r^2}$

$$\boxed{E = \frac{k}{2\varepsilon_0}\frac{R^2}{r^2}, \quad R < r < a}$$

$$\boxed{a < r < b \qquad E = 0}$$

$r > b$ $\qquad E = \frac{Q_{sph}}{4\pi\varepsilon_0 r^2} = \frac{k}{2\varepsilon_0}\frac{R^2}{r^2}$

$$\boxed{E = \frac{k}{2\varepsilon_0}\frac{R^2}{r^2}, \quad r > b}$$

(c) $$V = -\int_{\infty}^{0} E\,dr = -\left[\int_{\infty}^{b} E\,dr + \int_{b}^{a} \underbrace{E}_{0}\,dr + \int_{a}^{R} E\,dr + \int_{R}^{0} E\,dr\right]$$

$$V = -\left[\frac{k}{2\varepsilon_0} R^2 \int_{\infty}^{b} \frac{1}{r^2}\,dr + \frac{k}{2\varepsilon_0} R^2 \int_{a}^{R} \frac{1}{r^2}\,dr + \int_{R}^{0} \frac{k}{2\varepsilon_0}\,dr\right]$$

$$V = -\left[-\frac{k}{2\varepsilon_0}\frac{R^2}{r}\Big|_{\infty}^{b} - \frac{k}{2\varepsilon_0}\frac{R^2}{r}\Big|_{a}^{R} + \frac{k}{2\varepsilon_0} r\Big|_{R}^{0}\right]$$

$$\boxed{V = \frac{k}{2\varepsilon_0}\frac{R^2}{b} + \frac{kR^2}{2\varepsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right) + \frac{k}{2\varepsilon_0} R}$$

(d) The outer shell is grounded
$\Rightarrow$ $V(b) = 0$, there is no charge on the outer surface of the shell and the electric field outside the shell ($r > b$) is 0.

$$V(0) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{l} = -\left[\underbrace{\int_{\infty}^{b} E\,dr}_{0} + \underbrace{\int_{b}^{a} E\,dr}_{0} + \int_{a}^{R} E\,dr + \int_{R}^{0} E\,dr\right]$$

$$V(0) = -\left[\int_{a}^{R} E\,dr + \int_{R}^{0} E\,dr\right] = \frac{kR^2}{2\varepsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right) + \frac{k}{2\varepsilon_0} R$$

$$\boxed{V(0) = \frac{kR^2}{2\varepsilon_0}\left(\frac{2}{R} - \frac{1}{a}\right)}$$