

## Solution to problem set # 12

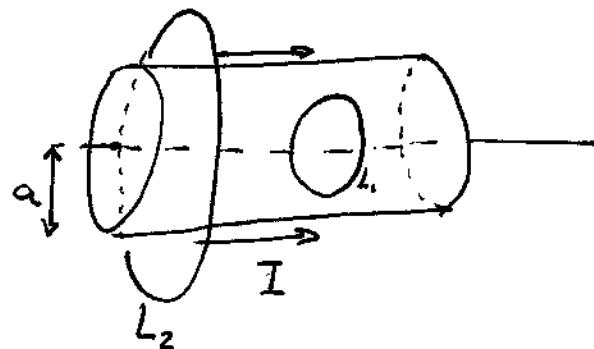
### Problem # 1 (book 5.13)

I - current

a - radius

(a) the current is on the surface only.

$$B_{in} = ? \quad B_{out} = ?$$



To find the field inside we select an Amperian loop  $L_1$  with radius  $s < a$

$$\oint_{L_1} \vec{B}_{in} \cdot d\vec{l} = \mu_0 I_{enc}$$

$I_{enc} = 0$  since all the current is on the surface of the wire.

$$\Rightarrow B_{in} (2\pi s) = 0$$

$$\Rightarrow \boxed{B_{in} = 0}$$

To find the field outside, we select an Amperian loop  $L_2$  with radius  $s > a$

$$\oint_{L_2} \vec{B}_{out} \cdot d\vec{l} = \mu_0 I_{enc}, \quad I_{enc} = I$$

$$B_{out} (2\pi s) = \mu_0 I$$

$$\boxed{B_{out} = \mu_0 \frac{I}{2\pi s} \hat{\phi}}$$

(b)  $|\vec{J}| = ks$ , where  $k$  is a constant and  $s$  is distance from the axis.

$$\underbrace{B_{in}=? \quad B_{out}=?}_{\text{To find } B_{in} \text{ and } B_{out}}$$

To find  $B_{in}$  and  $B_{out}$  we use the same Amperian loops as in part (a)

$$\oint \vec{B}_{in} \cdot d\vec{l} = \mu_0 I_{enc}$$

L<sub>1</sub>

$$I_{enc} = \int \vec{J} \cdot d\vec{A} = \int J dA \quad \text{where } \int dA \text{ is the area}$$

$$|\vec{J}| = ks$$

of the cross section  
of the wire

$$I_{enc} = \int ks' d(\pi s'^2) = \int_{s=0}^s ks' (2\pi s') ds' = 2\pi k \int_0^s s'^2 ds' = \frac{2\pi k}{3} s^3$$

$$I_{enc} = \frac{2\pi}{3} ks^3$$

$$\Rightarrow B_{in} 2\pi s = \mu_0 \frac{2\pi}{3} ks^3$$

$$B_{in} = \frac{\mu_0}{3} ks^2 \quad \text{How much is } k?$$

$$I_{total} = I = \int_A \vec{J} \cdot d\vec{A} = \int_{s=0}^a ks' 2\pi s' ds' = \frac{2\pi}{3} k a^3$$

$$\frac{2\pi}{3} k a^3 = I \Rightarrow k = I \frac{3}{2\pi} \frac{1}{a^3}$$

$$\Rightarrow B_{in} = \mu_0 \frac{I s^2}{2\pi a^3}$$

For outside we have

$$\oint \vec{B}_{\text{out}} \cdot d\vec{l} = \mu_0 I_{\text{enc}} - \mu_0 I$$

$L_2$

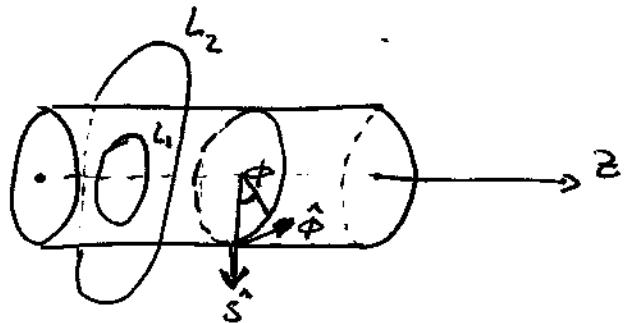
$$B_{\text{out}} \cdot 2\pi s = \mu_0 I$$

$$\boxed{B_{\text{out}} = \mu_0 \frac{I}{2\pi s}}$$

Problem #2 (book 6.8)

$$\begin{aligned} R &= \text{radius} \\ M &= ks^2 \hat{\phi} \end{aligned}$$

$$\overrightarrow{B}_{\text{in}} = ? \quad \overrightarrow{B}_{\text{out}} = ?$$



The magnetization  $\vec{M}$  results in bound currents  $\vec{K}_b$  and  $\vec{J}_b$ , and they ~~not~~ result in a magnetic field, with a vector potential  $\vec{A}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \int \frac{\vec{K}_b(\vec{r}')}{r} da'$$

$$\vec{K}_b = \vec{M} \times \hat{n}, \quad \hat{n} = \hat{s} \quad (\text{cylindrical coordinates})$$

$$\vec{K}_b = ks^2 \hat{\phi} \times \hat{s} = -ks^2 \hat{z} = -kR^2 \hat{z} \quad (s=R)$$

$$\boxed{\vec{K}_b = -kR^2 \hat{z}}$$

$$\vec{J}_b = \nabla \times \vec{M}_b = \frac{1}{s} \frac{\partial}{\partial s} (s K s^2) \hat{z} \quad (\text{in cylindrical coordinates})$$

$$\boxed{\vec{J}_b = 3 K s \hat{z}}$$

Note that  $\vec{K}_b$  and  $\vec{J}_b$  are in opposite directions and are closing the current loop.

Inside the cylinder

$$\oint_{L_1} \vec{B}_{in} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int_{\text{cross section}} \vec{J}_b \cdot d\vec{a} = \mu_0 \int_{s=0}^s 3 K s' 2\pi s' ds'$$

$$B_{in} 2\pi s = \mu_0 K 3 (2\pi) \int_{s'=0}^s s'^2 ds' = 2\pi \mu_0 K s^3$$

$$\boxed{B_{in} = \mu_0 K s^2 \hat{\phi}}$$

$$\oint_{L_2} \vec{B}_{out} \cdot d\vec{l} = \mu_0 I_{enc}$$

The enclosed current now is the total current on the cylinder

$$I_{total} = \underbrace{\int_A \vec{J}_b \cdot d\vec{a}}_{\text{current through the crosssection}} + \underbrace{\int \vec{K}_b \cdot d\vec{l}}_{\text{current on the surface}}$$



$$I_{\text{total}} = \int_{S=0}^R 3KS (2\pi s) ds + \int_0^L -kR^2 dl = \\ = 2\pi k R^3 - KR^2 L = 2\pi k R^3 - kR^2 (2\pi R) = 0$$

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$$\underline{I_{\text{total}} = 0 = I_{\text{enc}}}$$

$$\Rightarrow |\vec{B}_{\text{out}}| 2\pi S = 0$$

$$\boxed{B_{\text{out}} = 0}$$

This type of magnetization results in no magnetic field outside of the magnet.

### Problem #3 (book 6.10)

iron rod

L - length

a - side of square

square cross section

$\mu$

w - gap width,  $w \ll a \ll L$

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$\vec{B} = ?$

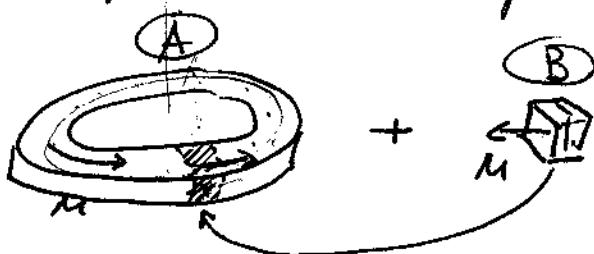
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The magnetization is uniform throughout the rod.

$$\vec{M} = |M| \hat{\phi}$$

There is no volume bound current  $J_b$  (and no free current). The only current is the surface bound current  $K_b$ .

We can present the field in the gap as a superposition of the field resulting from a full torus and the field from a short section (B) with an opposite magnetization.



$$\vec{B} = \vec{B}_A + \vec{B}_B$$

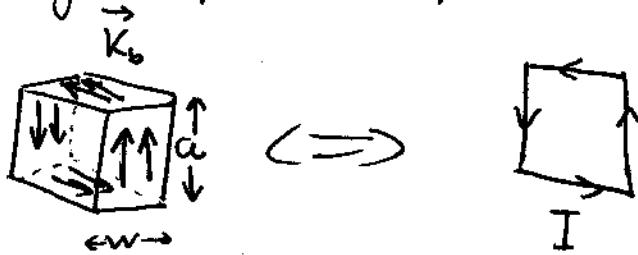
The field in the torus is

$$\oint B_A \cdot d\ell = \mu_0 I_{\text{enc}}$$

or  $\oint H_A \cdot d\ell = \mu_0 I_{\text{free}} = 0 \Rightarrow |\vec{H}_A| = 0 = \frac{1}{\mu_0} \vec{B} - \vec{M}$

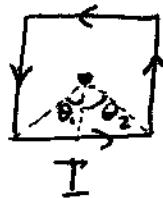
$$B_A = \mu_0 M \rightarrow \text{this is the magnetic field anywhere } \underline{\text{in}} \text{ the torus.}$$

The magnetic field from the small section is equivalent to a field from a square loop



$$I = K_b w = M w$$

The magnetic field in the center of a square loop is the sum from the contributions to the magnetic field from each side of the loop.



$$B_B = \sum_{i=1}^4 B_i$$

Using the formula, we that we derived in class and is also in the book

$$|B_i| = \frac{\mu_0 I}{4\pi s} (\sin \theta_1, -\sin \theta_2) \quad \text{where } s = \frac{a}{2} \text{ is}$$

the distance to the wire,  $\theta_1 = \theta_2 = 45^\circ$

$$|B_i| = \frac{\mu_0 I}{2\pi a} \left( 2 \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} \frac{\mu_0 I}{\pi a}$$

$$B_B = 4 B_i = 2\sqrt{2} \frac{\mu_0 I}{\pi a} = \frac{\mu_0 M w}{\pi a} 2\sqrt{2}$$

$$\Rightarrow B = B_A - B_B = \mu_0 M \left[ 1 - \frac{2\sqrt{2}}{\pi} \frac{w}{a} \right]$$