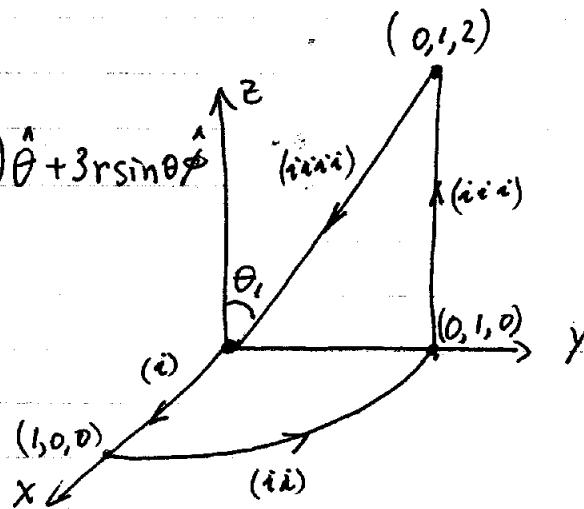


Solution to problem set #3

Problem 1

$$\vec{v} = (r^2 \cos^2 \theta) \hat{r} - (2r \cos \theta \sin \theta) \hat{\theta} + 3r \sin \theta \hat{\phi}$$

@ $\oint \vec{v} \cdot d\vec{l} = ?$



Separate the path in segments. (i) (ii) (iii) (iv)

segment (i) $d\vec{l} = dr \hat{r}$ (spherical coordinates)
 $\phi = 0, \theta = \frac{\pi}{2}$

\Rightarrow along this segment $\cos \theta = 0, \sin \theta = 1 \Rightarrow \vec{v} = 3r \hat{\phi}$

$\int_i \vec{v} \cdot d\vec{l} = 0$ since $\vec{v} \cdot d\vec{l} = 3r dr (\hat{r} \cdot \hat{\phi}) = 0$

segment (ii) $d\vec{l} = r d\phi \hat{\phi}$ and $r=1, \theta = \frac{\pi}{2} \Rightarrow \sin \theta = 1, \cos \theta = 0$
 ϕ varies from 0 to $\frac{\pi}{2}$

\Rightarrow along this segment $\vec{v} = 3r \hat{\phi} = 3 \hat{\phi}$

$\int_{ii} \vec{v} \cdot d\vec{l} = \int 3 \hat{\phi} \cdot \hat{\phi} r d\phi = 3 \int_0^{\frac{\pi}{2}} d\phi = \frac{3}{2}\pi$

segment (iii) Along this segment $y = \text{const} = r \sin \theta = 1$
where $\phi = \frac{\pi}{2}$

$\Rightarrow r = \frac{1}{\sin \theta}; dr = \frac{d}{d\theta} \left(\frac{1}{\sin \theta} \right) d\theta = -\frac{\cos \theta}{\sin^2 \theta} d\theta$

$$d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \phi \hat{\phi}$$

Along this path \vec{v} has all components.

$$\vec{v} \cdot d\vec{l} = r^2 \cos^2 \theta dr + (-2r \cos \theta \sin \theta) (r d\theta)$$

substitute "r" and "dr" in the expression above

$$\vec{v} \cdot d\vec{l} = \frac{\cos^2 \theta}{\sin^2 \theta} \left(-\frac{\cos \theta}{\sin^2 \theta} \right) d\theta - 2 \frac{\cos \theta}{\sin \theta} d\theta$$

$$\vec{v} \cdot d\vec{l} = -\frac{\cos^3 \theta}{\sin^4 \theta} d\theta - 2 \frac{\cos \theta}{\sin \theta} d\theta$$

Now integrate along path (iii)

$$\int_{\text{iii}} \vec{v} \cdot d\vec{l} = - \int_{\text{iii}} \frac{\cos^3 \theta}{\sin^4 \theta} d\theta - 2 \int_{\text{iii}} \frac{\cos \theta}{\sin \theta} d\theta$$

Along this path θ varies from $\frac{\pi}{2}$ to θ_1 , where

$$\theta_1 = 90^\circ - \tan^{-1} 2 = 90^\circ - 63.4 = 26.6^\circ \approx 26^\circ$$

$$\begin{aligned} \int_{\text{iii}} \vec{v} \cdot d\vec{l} &= - \int_{\frac{\pi}{2}}^{\theta_1} \frac{1 - \sin^2 \theta}{\sin^4 \theta} d(\sin \theta) - 2 \int_{\frac{\pi}{2}}^{\theta_1} \frac{d(\sin \theta)}{\sin \theta} = \\ &= - \int_{\frac{\pi}{2}}^{\theta_1} \frac{1}{\sin^3 \theta} d(\sin \theta) + \int_{\frac{\pi}{2}}^{\theta_1} \frac{1}{\sin^2 \theta} d(\sin \theta) - 2 \int_{\frac{\pi}{2}}^{\theta_1} \frac{1}{\sin \theta} d(\sin \theta) \\ &= \left[-\frac{1}{(3)\sin^3 \theta} + \frac{1}{(-1)\sin \theta} - 2 \ln(\sin \theta) \right] \Big|_{\frac{\pi}{2}}^{\theta_1} \end{aligned}$$

$$\sin \theta_1 \approx 0.45, \quad \sin \frac{\pi}{2} = 1$$

$$\begin{aligned} \int_{iii} \vec{v} \cdot d\vec{l} &= \left[\frac{1}{3 \times 0.45^3} - \frac{1}{0.45} - 2 \ln(0.45) \right] - \left[\frac{1}{3} - 1 \right] \\ &= [3.66 - 2.22 + 2 \times 0.8] - [-0.67] = 3.04 + 0.67 \end{aligned}$$

$$\int_{iii} \vec{v} \cdot d\vec{l} = 3.71$$

Segment (iv)

$$d\vec{l} = dr \hat{r}, \quad \varPhi = \frac{\pi}{2}, \quad \theta = \theta_1, \quad "r" \text{ varies from } \sqrt{5} \text{ to } 0$$

$$\Rightarrow \vec{v} \cdot d\vec{l} = r^2 \cos^2 \theta dr$$

$$\int_{iv} \vec{v} \cdot d\vec{l} = \int_{r=\sqrt{5}}^0 r^2 \cos^2 \theta_1 dr = \cos^2 \theta_1 \frac{r^3}{3} \Big|_{\sqrt{5}}^0 = -0.8 \times \frac{11.18}{3} = -2.98$$

$$\Rightarrow \oint \vec{v} \cdot d\vec{l} = 0 + \frac{3}{2} \pi + 3.71 - 2.98 = 5.44$$

$\oint \vec{v} \cdot d\vec{l} = 5.44$

(b) Stokes' theorem:

$$\oint \vec{V} \cdot d\vec{l} = \iint_{\text{enclosed surface}} (\nabla \times \vec{V}) \cdot d\vec{a}$$

path

inclosed
surface

Let's calculate $\nabla \times \vec{V}$ using spherical coordinates.

$$\nabla \times \vec{V} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta V_\phi) - \frac{\partial V_\theta}{\partial \phi} \right] \hat{r} +$$

$$+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r} (r V_\phi) \right] \hat{\theta} +$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r V_\theta) - \frac{\partial V_r}{\partial \theta} \right] \hat{\phi}$$

$$\nabla \times \vec{V} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (3r \sin^2 \theta) - 0 \right] \hat{r} + \frac{1}{r} \left[-\frac{\partial}{\partial r} (3r^2 \sin \theta) \right] \hat{\theta} +$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (-2r^2 \cos \theta \sin \theta) - \frac{\partial}{\partial \theta} (r^2 \cos^2 \theta) \right] \hat{\phi}$$

$$\nabla \times \vec{V} = 6 \cos \theta \hat{r} - 6 \sin \theta \hat{\theta} + [-2 \cos \theta \sin \theta + 2r \cos \theta \sin \theta] \hat{\phi}$$

$$\nabla \times \vec{V} = 6 \cos \theta \hat{r} - 6 \sin \theta \hat{\theta} + 2(r-2) \cos \theta \sin \theta \hat{\phi}$$

We separate the total surface integral in two parts:

I the integral over the surface in (x, y) plane

II the integral over the surface in (y, z) plane.

For surface I $d\vec{a} = dl_r dl_\theta \hat{\theta} = (dr)(r d\phi) \hat{\theta}$

$$d\vec{a} = r dr d\phi \hat{\theta}$$

$$\Rightarrow I = \int_{\text{I}} (\nabla \times \vec{v}) \cdot d\vec{a} = \int_{r=0}^1 \int_{\theta=0}^{\pi/2} (-6 \sin \theta) r dr d\phi = \\ = -6 \times \frac{\pi}{2} \times \frac{r^2}{2} \Big|_0^1 = -\frac{3\pi}{2}$$

For surface II $d\vec{a} = dl_r dl_\theta \hat{\phi} = (dr)(r d\theta) \hat{\phi}$

$$d\vec{a} = r dr d\theta \hat{\phi}$$

$$\text{II} = \int_{\text{II}} (\nabla \times \vec{v}) \cdot d\vec{a} = \int_{\theta=\pi/2}^{\theta_1} \int_{r=0}^{r_1 \sin \theta} 2(r-2) \cos \theta \sin \theta r dr d\theta = \\ = \int_{\theta=\pi/2}^{\theta_1} \cos \theta \sin \theta \left(\int_{r=0}^{\sin \theta} (2r^2 - 4r) dr \right) d\theta = \int_{\theta=\pi/2}^{\theta_1} \cos \theta \sin \theta \left(\frac{2r^3}{3} - 2r^2 \right) \Big|_0^{\sin \theta} d\theta = \\ = \int_{\theta=\pi/2}^{\theta_1} \frac{2}{3} \frac{\cos \theta}{\sin^2 \theta} d\theta - \int_{\theta=\pi/2}^{\theta_1} 2 \frac{\cos \theta}{\sin \theta} d\theta = \int_{\theta=\pi/2}^{\theta_1} \frac{2}{3} \frac{d(\sin \theta)}{\sin^2 \theta} - 2 \int_{\theta=\pi/2}^{\theta_1} \frac{d(\sin \theta)}{\sin \theta} = \\ = -\frac{2}{3} \cdot \frac{1}{\sin \theta} \Big|_{\theta=\pi/2}^{\theta_1} - 2 \ln(\sin \theta) \Big|_{\theta=\pi/2}^{\theta_1} = \\ = \frac{2}{3} \left(1 - \frac{1}{\sin \theta_1} \right) - 2 \ln(\sin \theta_1) ; \sin \theta_1 = \frac{1}{\sqrt{5}}$$

$$\text{II} = -0.824 + 1.609 = 0.785 \Rightarrow \boxed{I + \text{II} = \frac{3\pi}{2} + 0.785 = 5.497}$$

Problem 2

spherical shell

a

b

$$g(r, \theta, \phi) = g_0 \frac{r}{b} \cos^2 \theta$$

$$g_0 = \text{const.} > 0$$

$$\underbrace{Q = ?}_{\text{?}}$$

$$Q = \int_V dq = \int_V g(r, \theta, \phi) dV ,$$

where $dq = g(r, \theta, \phi) dV$ is the charge of a very small volume dV . The integration is over the volume of the shell.

This problem has spherical symmetry and it is useful to use spherical coordinates.

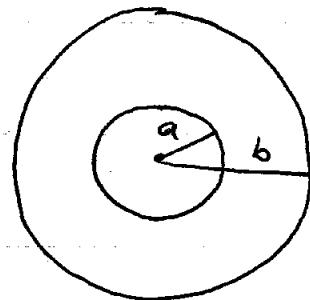
$$\Rightarrow dV = r^2 \sin \theta dr d\theta d\phi$$

$$Q = \int_{r=a}^b \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} g(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi = \int_{r=a}^b \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} g_0 \frac{r^3}{b} \cos^2 \theta \sin \theta dr d\theta d\phi$$

$$Q = \frac{g_0}{b} 2\pi \left[\frac{r^4}{4} \right]_a^b \int_{\theta=0}^{\pi} \cos^2 \theta \sin \theta d\theta = -\frac{\pi}{2} \frac{g_0}{b} (b^4 - a^4) \int_{\theta=0}^{\pi} \cos^3 \theta d\cos \theta$$

$$Q = -\frac{\pi}{2} \frac{g_0}{b} (b^4 - a^4) \left[\frac{\cos^3 \theta}{3} \right]_{\theta=0}^{\pi} = -\frac{\pi}{2} \frac{g_0}{b} (b^4 - a^4) \frac{1}{3} (-1 - 1)$$

$$Q = \frac{\pi}{3} g_0 \frac{b^4 - a^4}{b}$$



Problem 3

For any a and b find $\int_{-\infty}^a x^2 \delta(x-b) dx$

$$\int_{-\infty}^a x^2 \delta(x-b) dx = \begin{cases} 0, & \text{for } b > a \\ b^2, & \text{for } b < a \end{cases}$$

Problem 4

(a) $g(\vec{r}) = m \delta^3(\vec{r} - \vec{r}') = \begin{cases} 0, & \text{for } \vec{r} \neq \vec{r}' \\ \infty, & \text{for } \vec{r} = \vec{r}' \end{cases}$

The integral of $g(\vec{r})$ over the entire space should give me the mass of the particle. We have to check.

$$I = \int_{\text{all space}} g(\vec{r}) d\tau = \int_{\text{all space}} m \delta^3(\vec{r} - \vec{r}') d\tau = m \underbrace{\int_{\text{all space}} \delta^3(\vec{r} - \vec{r}') d\tau}_1$$

$$I = m \quad \checkmark$$

(b) The mass density function should describe two point masses.

$$g(\vec{r}) = m \delta^3(\vec{r}) + m \delta^3(\vec{r} - \vec{a}) = \begin{cases} 0, & \text{for } \vec{r} \neq 0, \vec{r} \neq \vec{a} \\ \infty, & \text{for } \vec{r} = 0 \\ \infty, & \text{for } \vec{r} = \vec{a} \end{cases}$$

$$\int_{\text{all space}} g(\vec{r}) d\tau = \int_{\text{all space}} m \delta^3(\vec{r}) d\tau + \int_{\text{all space}} m \delta^3(\vec{r} - \vec{a}) d\tau = m + m = 2m$$

(c)

We consider a infinitesimally thin spherical shell of radius R , and total mass M .

The mass density should be "0" everywhere but at the surface of the shell.

$$g(\vec{r}) = g_0 \delta(r-R) = \begin{cases} 0, & r \neq R \\ \infty, & r=R \end{cases}$$

How much is g_0 ? This we get from the constraint that

$$\underbrace{\int_{\text{all space}} g_0 \delta(r-R) dr}_{\text{all space}} = M$$

note that this is the 1Dimensional δ , and not δ^3

$$\begin{aligned} \int_{\text{all space}} g_0 \delta(r-R) dr &= \iiint_{\substack{r=0 \\ \theta=0 \\ \phi=0}}^{r=\infty} g_0 \delta(r-R) r^2 \sin\theta dr d\theta d\phi = \\ &= g_0 2\pi \times 2 \int_{r=0}^{\infty} r^2 \delta(r-R) dr = g_0 4\pi R^2 \end{aligned}$$

$$\Rightarrow g_0 4\pi R^2 = M \quad g_0 = \frac{M}{4\pi R^2}$$

$$\Rightarrow \boxed{g(\vec{r}) = \frac{M}{4\pi R^2} \delta(r-R)}$$