

Solution to problem set #9

Problem 1:

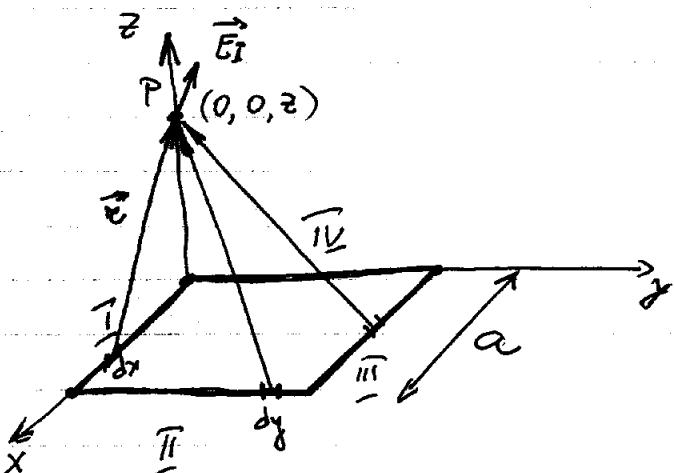
λ - linear charge density

a - side of the square

$\vec{r} = (0, 0, z)$ position of the point.

$$\vec{E} = ?$$

The electric field \vec{E} at point P is a superposition of the electric field created by the separate segments of the loop (I, II, III, IV)



$$\vec{E} = \vec{E}_I + \vec{E}_{II} + \vec{E}_{III} + \vec{E}_{IV}$$

Let's calculate first \vec{E}_I

$$\vec{E}_I = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{\epsilon}}{r^2} dq , \quad dq = \lambda dx , \quad \vec{r} = \frac{\vec{r}}{r}$$

$$\vec{E}_I = \frac{1}{4\pi\epsilon_0} \int_{x=0}^a \frac{\vec{\epsilon}}{r^3} dx , \quad r = (x^2 + z^2)^{1/2} , \quad \vec{\epsilon} = -x\hat{x} + z\hat{z}$$

$$x=0$$

$$\vec{E}_I = \frac{\lambda}{4\pi\epsilon_0} \int_{x=0}^a \left(-\frac{x}{(x^2 + z^2)^{3/2}} \hat{x} + \frac{z}{(x^2 + z^2)^{3/2}} \hat{z} \right) dx$$

$$\vec{E}_I = \frac{\lambda}{4\pi\epsilon_0} \left[-\hat{x} \int_{x=0}^a \frac{x}{(x^2 + z^2)^{3/2}} dx + \hat{z} z \int_{x=0}^a \frac{1}{(x^2 + z^2)^{3/2}} dx \right]$$

$$\vec{E}_I = \frac{\lambda}{4\pi\epsilon} \left[-\hat{x} \frac{1}{2} \int_{x=0}^a \frac{dx(x^2+z^2)}{(x^2+z^2)^{3/2}} + \hat{z} z \int_{x=0}^a \frac{dx}{(x^2+z^2)^{3/2}} \right]$$

$$\vec{E}_I = \frac{\lambda}{4\pi\epsilon} \left[\hat{x} (x^2+z^2)^{-1/2} \Big|_{x=0}^a + \hat{z} z \left(\frac{x}{z^2} (x^2+z^2)^{-1/2} \right) \Big|_{x=0}^a \right]$$

(I) $\vec{E}_I = \frac{-\lambda}{4\pi\epsilon} \left(\frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}} \right) \hat{x} + \frac{\lambda}{4\pi\epsilon} \left(\frac{a}{z} \frac{1}{(a^2+z^2)^{1/2}} \right) \hat{z}$

Let's calculate \vec{E}_{II}

$$\vec{E}_{II} = \frac{1}{4\pi\epsilon} \int \frac{\vec{e}}{r^2} dq, \quad dq = \lambda dy \quad \vec{e} = \frac{\vec{r}}{r}$$

$$\vec{E}_{II} = \frac{\lambda}{4\pi\epsilon} \int_{y=0}^a \frac{(-a\hat{x}-y\hat{y}+z\hat{z})}{(a^2+y^2+z^2)^{3/2}} dy, \quad \vec{e} = (a^2+y^2+z^2)^{1/2}$$

$$\vec{E}_{II} = \frac{\lambda}{4\pi\epsilon} \left[-a\hat{x} \int_{y=0}^a \frac{dy}{(a^2+z^2+y^2)^{3/2}} - \hat{y} \int_{y=0}^a \frac{y dy}{(a^2+z^2+y^2)^{3/2}} + z\hat{z} \int_{y=0}^a \frac{dy}{(a^2+z^2+y^2)^{3/2}} \right]$$

$$\vec{E}_{II} = \frac{\lambda}{4\pi\epsilon} \left[\hat{x} \frac{(-a)}{a^2+z^2} \frac{y}{(a^2+z^2+y^2)^{1/2}} \Big|_{y=0}^a + \hat{y} \frac{1}{(a^2+z^2+y^2)^{1/2}} \Big|_{y=0}^a + \hat{z} \frac{z}{a^2+z^2} \frac{y}{(a^2+z^2+y^2)^{1/2}} \Big|_{y=0}^a \right]$$

(II) $\vec{E}_{II} = \frac{-\lambda}{4\pi\epsilon} \frac{a^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} \hat{x} - \frac{\lambda}{4\pi\epsilon} \left(\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{(2a^2+z^2)^{1/2}} \right) +$
 $+ \frac{\lambda}{4\pi\epsilon} \frac{za}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} \hat{z}$

-3-

Let's calculate \vec{E}_{III}

$$\vec{E}_{\text{III}} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{e}}{r^2} dq \quad , \quad dq = \lambda dx, \quad \hat{e} = \frac{\vec{r}}{r}$$

$$\vec{E}_{\text{III}} = \frac{\lambda}{4\pi\epsilon_0} \int_0^a \frac{(-x\hat{x} - a\hat{y} + z\hat{z}) dx}{(x^2 + a^2 + z^2)^{3/2}} \quad \left| \begin{array}{l} \vec{r} = -x\hat{x} - a\hat{y} + z\hat{z} \\ r = (x^2 + a^2 + z^2)^{1/2} \end{array} \right.$$

$$\vec{E}_{\text{III}} = \frac{\lambda}{4\pi\epsilon_0} \left[-\hat{x} \int_{x=0}^a \frac{x dx}{(x^2 + a^2 + z^2)^{3/2}} - a\hat{y} \int_{x=0}^a \frac{dx}{(x^2 + a^2 + z^2)^{3/2}} + z\hat{z} \int_{x=0}^a \frac{dx}{(x^2 + a^2 + z^2)^{3/2}} \right]$$

$$\vec{E}_{\text{III}} = \frac{\lambda}{4\pi\epsilon_0} \left[+\hat{x} \left. \frac{1}{\sqrt{x^2 + a^2 + z^2}} \right|_{x=0}^a - \hat{y} \left. \frac{a}{(a^2 + z^2)} \right|_{x=0}^a - \left. \frac{x}{(x^2 + a^2 + z^2)^{1/2}} \right|_{x=0}^a + \hat{z} \left. \frac{z}{a^2 + z^2} \right|_{x=0}^a \right]$$

$$\text{(III)} \quad \vec{E}_{\text{III}} = -\frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{2a^2 + z^2}} \right) \hat{x} - \frac{a^2}{a^2 + z^2} \frac{1}{(2a^2 + z^2)^{1/2}} \hat{y} + \frac{\lambda}{4\pi\epsilon_0} \frac{za}{a^2 + z^2} \frac{1}{\sqrt{2a^2 + z^2}} \hat{z}$$

Let's calculate IV segment: \vec{E}_{IV}

One can argue that

$$\vec{E}_{\text{IV}} = E_{\text{IV},x} \hat{x} + E_{\text{IV},y} \hat{y} + E_{\text{IV},z} \hat{z}$$

and because of symmetry

$$E_{\text{IV},x} = 0, \quad E_{\text{IV},y} = E_{\text{I},x}, \quad E_{\text{IV},z} = E_{\text{I},z}$$

- 4 -

IV $\vec{E}_{IV} = -\frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}} \right) \hat{y} + \frac{\lambda}{4\pi\epsilon_0} \left(\frac{a}{z} \frac{1}{(a^2+z^2)^{1/2}} \right) \hat{z}$

Now we sum the corresponding components from I, II, III and IV

$$E_x = E_{1x} + E_{2x} + E_{3x} + E_{4x}$$

$$E_x = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}} + \frac{a^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} + \frac{1}{(a^2+z^2)^{1/2}} - \right.$$

$$\left. - \frac{1}{(2a^2+z^2)^{1/2}} \right] = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{z} + \frac{1}{(2a^2+z^2)^{1/2}} \frac{a^2-a^2-z^2}{a^2+z^2} \right]$$

✓ $E_x = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{z} - \frac{z^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} \right]$

$$E_y = E_{1y} + E_{2y} + E_{3y} + E_{4y}$$

$$E_y = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{(2a^2+z^2)^{1/2}} + \frac{a^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} + \right.$$

$$\left. + \frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}} \right] = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{z} - \frac{z^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} \right]$$

✓ $E_y = -\frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{z} - \frac{z^2}{a^2+z^2} \frac{1}{(2a^2+z^2)^{1/2}} \right]$

(it is equal to the E_x)

$$E_z = E_{1z} + E_{2z} + E_{3z} + E_{4z}$$

$$E_z = \frac{2\lambda}{4\pi\epsilon_0} \left[\frac{a}{z} \frac{1}{(a^2+z^2)^{1/2}} + \frac{2a}{(a^2+z^2)^{1/2}} \frac{1}{(2a^2+z^2)^{1/2}} \right]$$

✓ $E_z = \frac{\lambda}{2\pi\epsilon_0} \frac{a}{z} \frac{1}{(a^2+z^2)^{1/2}} \left[1 + \frac{z^2}{(2a^2+z^2)^{1/2}} \right]$

Problem 2 :

- a - inner radius
- b - outer radius
- q - total charge

$$\vec{E} = ?$$

There are more than one way to solve this problem. This is one of them.

$$\vec{E} = \int \frac{1}{4\pi\epsilon_0} \frac{\epsilon_0}{r^2} dq \quad dq = \sigma da$$

where σ is the surface charge density; $\sigma = \text{constant}$.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{e}_r}{r^3} \sigma da = \frac{\sigma}{4\pi\epsilon_0} \int \frac{\vec{e}_r}{(s^2 + z^2)^{3/2}} s ds d\phi$$

where $da = s ds d\phi$ in cylindrical coordinates

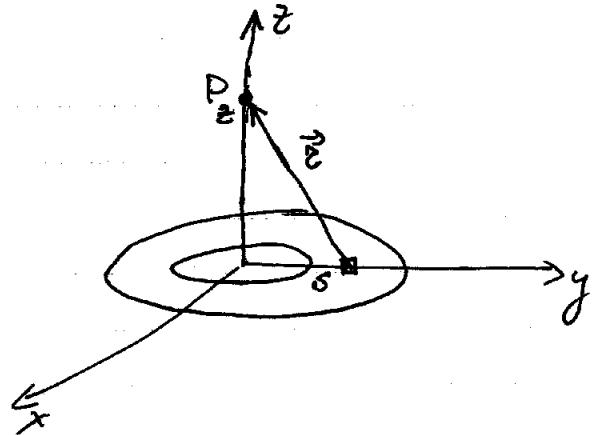
$$\vec{e}_r = s \hat{s} + 0 \hat{\phi} + z \hat{z}$$

$$\vec{E} = \frac{\sigma}{4\pi\epsilon_0} \iint_{s=a, \phi=0}^{s=b, \phi=2\pi} \frac{s^2 \hat{s}}{(s^2 + z^2)^{3/2}} ds d\phi + \frac{\sigma}{4\pi\epsilon_0} \iint_{s=a, \phi=0}^{s=b, \phi=2\pi} \frac{zs \hat{z}}{(s^2 + z^2)^{3/2}} ds d\phi$$

$$\vec{E} = \frac{\sigma}{4\pi\epsilon_0} \int_{s=a}^b \frac{s^2}{(s^2 + z^2)^{3/2}} ds \underbrace{\int_{\phi=0}^{2\pi} (\cos\phi \hat{x} + \sin\phi \hat{y}) d\phi}_{= 2\pi} +$$

$$+ \frac{\sigma z \hat{z}}{4\pi\epsilon_0} \int_{s=a, \phi=0}^{s=b, \phi=2\pi} \frac{s}{(s^2 + z^2)^{3/2}} ds d\phi = \frac{\sigma z}{2\epsilon_0} \hat{z} \int_{s=a}^b \frac{s ds}{(s^2 + z^2)^{3/2}}$$

$$\vec{E} = \frac{\sigma z}{2\epsilon_0} (-1) \left(\frac{1}{(s^2 + z^2)^{1/2}} \right) \Big|_{s=a}^b \quad \hat{z} = \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{(a^2 + z^2)^{1/2}} - \frac{1}{(b^2 + z^2)^{1/2}} \right] \hat{z}$$



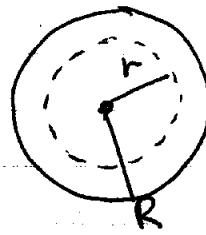
Problem 3

$$S = k\sigma$$

k = constant

R - radius

$$\vec{E}(r) = ? \text{ for } r < R$$



Select a Gauss surface : a sphere centered at the given charged sphere and a radius $r < R$. It is shown with dashed line.

From Gauss Law

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$Q_{\text{enc}} = \int_S S d\tau = \iiint_{V \text{ of the Gauss sphere}} (kr^3) r^2 \sin\theta dr' d\theta d\phi =$$

$$= 4\pi \int_{r'=0}^r kr'^3 dr' = 4\pi k \frac{r^4}{4} \Big|_0^r = \pi k r^4$$

$$Q_{\text{enc}} = \pi k r^4$$

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{a} = \frac{\pi k r^4}{\epsilon_0} = |\vec{E}| 4\pi r^2$$

$$|\vec{E}| = \frac{1}{4\epsilon_0} kr^2$$

$$\vec{E} = \frac{1}{4\epsilon_0} kr^2 \hat{r}$$

, $r < R$