

## Solution to problem set #7

### Problem 1

$x, y \rightarrow$  semi-infinite

grounded, conducting  
planes.

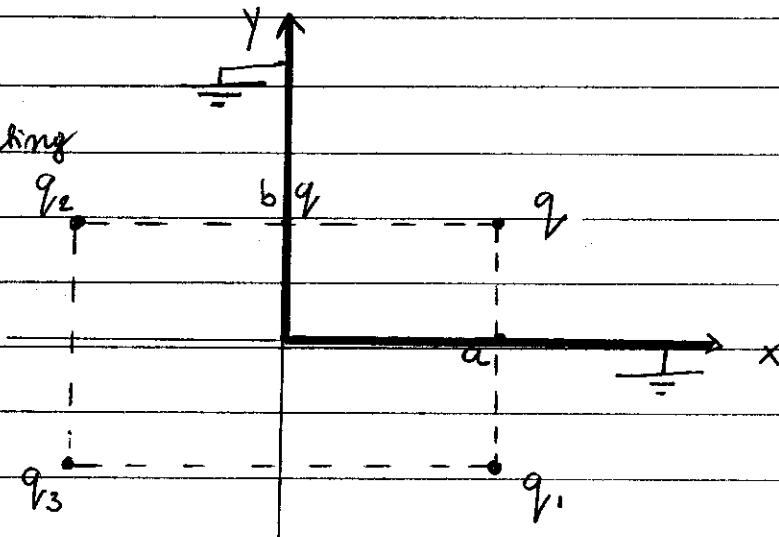
$q$  is at location  $\vec{r}$

$$\vec{r} = (a, b)$$

$$V(\vec{r}) = ?$$

for  $\vec{r}$  such that

$$r_x > 0, r_y > 0$$



We use the method of images.

The potential in the space between the two conducting and grounded plates is a result of the point charge  $q$  and the induced charges on the two metal plates.

The charge  $q_1$  induced on plate X is a mirror image of  $q$ .

$$q_1 = -q \text{ and is located at } \vec{r}_1'(a, -b)$$

In addition, the charge on the plate X is influenced by the charge distribution on plate y.

The charge  $q_2$  induced on plate y is

$$q_2 = q \text{ at } \vec{r}_2'(a, b)$$

In addition to  $q_2$ , the charge distribution on plate y is influenced by the induced charge on plate X.

The interaction between the charges on plate X and Y is represented by the charge  $q_3$ .  
 $q_3 = -q_2 = -q$ ,  $q_3$  is at  $r_3'(-a, -b)$

It is a mirror image of  $q_1$  with respect to the Y plate, it is also a mirror image of  $q_2$  with respect to the X plate.

The potential  $V(r)$  in the region  $r(x>0, y>0)$  is equivalent the potential of four point charges ( $q, q_1, q_2, q_3$ ). Note that in the region of interest ( $x>0, y>0$ ) there is only one point charge  $q$ . All imaginary charges ( $q_1, q_2, q_3$ ) are outside the region of interest.

$$V(r) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r} + \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} \right)$$

$$r = |\vec{r}| = \sqrt{(x-a)^2 + (y-b)^2 + z^2}, \quad \vec{r} = \vec{r} - \vec{r}'$$

$$r_1 = |\vec{r}_1| = \sqrt{(x-a)^2 + (y+b)^2 + z^2}, \quad \vec{r}_1 = \vec{r} - \vec{r}_1', \quad q_1 = -q$$

$$r_2 = |\vec{r}_2| = \sqrt{(x+a)^2 + (y-b)^2 + z^2}, \quad \vec{r}_2 = \vec{r} - \vec{r}_2', \quad q_2 = -q$$

$$r_3 = |\vec{r}_3| = \sqrt{(x+a)^2 + (y+b)^2 + z^2}, \quad \vec{r}_3 = \vec{r} - \vec{r}_3', \quad q_3 = q$$

$$V(r) = \frac{q}{4\pi\epsilon_0} \left[ \left( (x-a)^2 + (y-b)^2 + z^2 \right)^{-1/2} - \left( (x-a)^2 + (y+b)^2 + z^2 \right)^{-1/2} - \left( (x+a)^2 + (y-b)^2 + z^2 \right)^{-1/2} + \left( (x+a)^2 + (y+b)^2 + z^2 \right)^{-1/2} \right], \quad \text{for } x>0, y>0$$

(b) What is the force acting on  $q$ ?

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$F_x = F_{x_1} + F_{x_2} + F_{x_3} = 0 + \frac{1}{4\pi\epsilon_0} \frac{q(-q)}{(2a)^2} + \frac{1}{4\pi\epsilon_0} \frac{qq}{4(a^2+b^2)} \frac{a}{\sqrt{a^2+b^2}}$$

$$F_x = \frac{q^2}{16\pi\epsilon_0} \left[ \frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2} \right]$$

$$F_y = F_{y_1} + F_{y_2} + F_{y_3} = \frac{1}{4\pi\epsilon_0} \frac{q(-q)}{(2b)^2} + 0 + \frac{1}{4\pi\epsilon_0} \frac{qq}{4(a^2+b^2)} \frac{b}{\sqrt{a^2+b^2}}$$

$$F_y = \frac{q^2}{16\pi\epsilon_0} \left[ \frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2} \right]$$

$$|F| = \left( F_x^2 + F_y^2 \right)^{1/2}$$

(c) The energy required to bring the charge  $q$  from  $\infty$  to point  $P(a, b)$  can be calculated in two ways

i) Calculate  $\vec{E} = \nabla V$ , where  $V$  is taken from (a) and then calculate

$$W = \frac{\epsilon_0}{2} \int E^2 dV, \text{ where we integrate}$$

only over  $\frac{1}{4}$  of all space. ( $x \in (0, +\infty)$ ,  $y \in (0, +\infty)$ ,  $z \in (-\infty, +\infty)$ )

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2) The second method is by using the expression for the work done by an electric force  $\vec{F}_{el}$ .

$$W = - \int_{\text{el}} \vec{F}_{el} \cdot d\vec{l} \quad \text{and we use } \vec{F}_{el} \text{ calculated in (b)}$$

/ shall use the second method

$$W = - \int_{\text{el}} \vec{F}_{el} \cdot d\vec{l} = - \int_a^b F_x dx + \int_a^b F_y dy$$

$$W = + \int_a^{\infty} \frac{q^2}{16\pi\epsilon_0} \left[ \frac{x}{(x^2+b^2)^{3/2}} - \frac{1}{x^2} \right] dx +$$

$$+ \int_{\infty}^b \frac{q^2}{16\pi\epsilon_0} \left[ \frac{y}{(a^2+y^2)^{3/2}} - \frac{1}{y^2} \right] dy$$

Here we used the fact that the electrostatic force is conservative and the work done by it does not depend on the exact path taken by the charge to get from point  $(x=\infty, y=\infty)$  to  $(x=a, y=b)$

$$W = \frac{+q^2}{16\pi\epsilon_0} \left[ \int_a^{\infty} \frac{\frac{1}{2}dx^2}{(x^2+b^2)^{3/2}} - \int_{\infty}^a \frac{dx}{x^2} \right] +$$

$$+ \frac{q^2}{16\pi\epsilon_0} \left[ \int_{\infty}^b \frac{\frac{1}{2}dy^2}{(a^2+y^2)^{3/2}} - \int_{\infty}^b \frac{dy}{y^2} \right]$$

$$W = \frac{-q^2}{16\pi\epsilon_0} \left[ \frac{1}{\sqrt{x^2+b^2}} \Big|_{\infty}^a + \frac{1}{x} \Big|_{\infty}^a - \frac{1}{\sqrt{a^2+y^2}} \Big|_{\infty}^b + \frac{1}{y} \Big|_{\infty}^b \right]$$

$$W = \frac{-q^2}{16\pi\epsilon} \left[ -\frac{1}{\sqrt{a^2+b^2}} + \frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}} + \frac{1}{b} \right]$$

$$W = \frac{-q^2}{16\pi\epsilon} \left[ -\frac{2}{\sqrt{a^2+b^2}} + \frac{1}{a} + \frac{1}{b} \right]$$

$$W = \frac{-q^2}{16\pi\epsilon} \left[ \frac{1}{a} + \frac{1}{b} - \frac{2}{\sqrt{a^2+b^2}} \right]$$

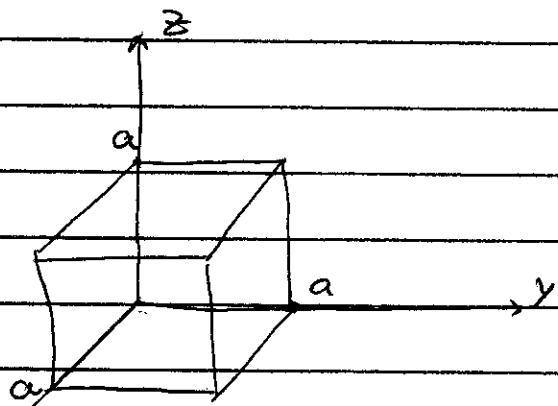
### Problem 2

cube with side a

5 of the sides are grounded

top side at  $V_0$

$V=?$  inside the box



The potential inside the box is given by the solution of Laplace's equation

$$\nabla^2 V = 0$$

We will use  $x, y, z$  coordinates with the following set of boundary conditions:

b.c.	$x=0$	$V=0$
	$x=a$	$V=0$
	$y=0$	$V=0$
	$y=a$	$V=0$
	$z=0$	$V=0$
	$z=a$	$V=V_0$

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$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (1)$$

Let's assume that  $V(x, y, z)$  is a separable function.

$$V(x, y, z) = X(x) Y(y) Z(z)$$

Substitute in (1)

$$Y(y) Z(z) \frac{\partial^2 X}{\partial x^2} + X(x) Z(z) \frac{\partial^2 Y}{\partial y^2} + X(x) Y(y) \frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z(z)} \frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} = - \frac{1}{Z(z)} \frac{\partial^2 Z}{\partial z^2} = -m^2$$

$$(2) \quad \frac{1}{Z(z)} \frac{\partial^2 Z}{\partial z^2} = m^2$$

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = - \frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} - m^2 = -k^2$$

$$(3) \quad \frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = -k^2$$

$$\frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} = -l^2 = -m^2 + k^2 \quad m^2 = l^2 + k^2 \quad (5)$$

$$(4) \quad \frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} = l^2$$

From equations (2), (3), (4) and (5) we obtain

$$X(x) = A \cos(kx) + B \sin(kx)$$

$$Y(y) = C \cos(ly) + D \sin(ly)$$

$$Z(z) = E e^{mz} + F e^{-mz}$$

and the general solution for the Laplace's equation is

$$(6) \quad V(x, y, z) = (A \cos(kx) + B \sin(kx))(C \cos_ly + D \sin_ly)(E e^{mz} + F e^{-mz})$$

Now we apply the boundary conditions

For  $x=0$   $V=0 = A(C \cos_ly + D \sin_ly)(E e^{mz} + F e^{-mz})$

and this should be true for any  $(y, z)$

$$\Rightarrow A=0$$

For  $y=0$   $V=0 = B \sin(kx), C(E e^{mz} + F e^{-mz})$

for any  $(x, z)$

$$\Rightarrow C=0$$

For  $z=0$   $V=0 = B \sin(kx) \sin(l y) (E + F)$

for any  $(x, y)$

$$\Rightarrow (E + F) = 0 \quad E = -F$$

The general solution (6) is now reduced to

$$(7) \quad V(x,y,z) = G \sin(kx) \sinh(ly) \sinh(mz)$$

where  $\sinh(mz) = \frac{e^{mz} - e^{-mz}}{2}$

and  $G = 2BDE = \text{constant}$

Now we apply the rest of the b.c.

at  $x=a$   $V=0 = G \sin(ka) \sinh(ly) \sinh(mz)$

for any  $(y, z)$

This requires that  $ka = 0, \pi, 2\pi, 3\pi, \dots = p\pi ; p=0, 1, 2, \dots$   
 $\Rightarrow k = \frac{p\pi}{a}, p=0, 1, 2, \dots$

at  $y=a$   $V=0 = G_p \sin\left(\frac{p\pi}{a}x\right) \sin(la) \sinh(mz)$

for any  $(x, z)$

This requires that  $la = 0, \pi, 2\pi, 3\pi, \dots = q\pi ; q=0, 1, 2, \dots$

$$\Rightarrow l = \frac{q\pi}{a}, q=0, 1, 2, \dots$$

The general solution (7) can be rewritten as

$$(8) \quad V(x,y,z) = \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} G_{pq} \sin\left(\frac{p\pi}{a}x\right) \sin\left(\frac{q\pi}{a}y\right) \sinh(mz)$$

at  $z=a$   $V=V_0$

$$(9) \quad V_0 = \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} G_{pq} \sin\left(\frac{p\pi}{a}x\right) \sin\left(\frac{q\pi}{a}y\right) \sinh(ma)$$

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Multiply both sides of (a) by  $\sin\left(\frac{P'\pi}{a}x\right) \sin\left(\frac{Q'\pi}{a}y\right)$  and integrate over x and y

$$\begin{aligned} & \iint_{0,0}^{a,a} V_0 \sin\left(\frac{P\pi}{a}x\right) \sin\left(\frac{Q\pi}{a}y\right) dx dy = \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} G_{pq} \underbrace{\sin\left(\frac{P\pi}{a}x\right) \sin\left(\frac{P\pi}{a}x\right) dx}_{\frac{a}{2} \text{ for } p'=p} \underbrace{\int \sin\left(\frac{Q\pi}{a}y\right) \sin\left(\frac{Q\pi}{a}y\right) dy}_{\frac{a}{2} \text{ for } q'=q} \\ & \quad 0 \text{ for } p' \neq p \quad 0 \text{ for } q' \neq q \\ \Rightarrow & \iint_{0,0}^{a,a} V_0 \sin\left(\frac{P\pi}{a}x\right) \sin\left(\frac{Q\pi}{a}y\right) dx dy = G_{pq} \left(\frac{a}{2}\right)^2 \operatorname{sh}(ma) \end{aligned}$$

$$G_{pq} = \left(\frac{2}{a}\right)^2 \frac{V_0}{\operatorname{sh}(ma)} \int_0^a \sin\left(\frac{P\pi}{a}x\right) dx \int_0^a \sin\left(\frac{Q\pi}{a}y\right) dy$$

if p or q is even  $\int_0^a \sin\left(\frac{P\pi}{a}x\right) dx = 0$  and  $\int_0^a \sin\left(\frac{Q\pi}{a}y\right) dy = 0$

if p and q is odd  $\int_0^a \sin\left(\frac{P\pi}{a}x\right) dx = \left[ -\frac{a}{P\pi} \cos\left(\frac{P\pi}{a}x\right) \right]_0^a = \frac{a}{P\pi}$   
 $\int_0^a \sin\left(\frac{Q\pi}{a}y\right) dy = \left[ -\frac{a}{Q\pi} \cos\left(\frac{Q\pi}{a}y\right) \right]_0^a = \frac{a}{Q\pi}$

$$\Rightarrow G_{pq} = \frac{4V_0}{pq\pi^2} \frac{1}{\operatorname{sh}(ma)} \quad \text{for } p \text{ and } q \text{ odd}$$

$$G_{pq} = 0 \quad \text{for } p \text{ or } q \text{ even}$$

$$\text{and } m^2 = k^2 + l^2 \Rightarrow m = \frac{\pi}{a} \sqrt{p^2 + q^2}$$

$$\Rightarrow G_{pq} = \frac{4V_0}{\pi^2 pq} \frac{1}{\operatorname{sh}(\pi\sqrt{p^2+q^2}z)}$$

The general solution is then given as

$$V(x, y, z) = \frac{4V_0}{\pi^2} \sum_{p=1,3,5,\dots} \sum_{q=1,3,5,\dots} \sin\left(\frac{p\pi}{a}x\right) \sin\left(\frac{q\pi}{a}y\right) \frac{\operatorname{sh}\left(\frac{\pi}{a}\sqrt{p^2+q^2}z\right)}{\operatorname{sh}(\pi\sqrt{p^2+q^2})}$$

### Problem 3

sphere, with radius R  
at  $r=R$   $V_0 = k \cos(3\theta)$

$V=?$  for  $r < R$  and  $r > R$ ;  $\phi=?$

We will solve the Laplace's equation in spherical coordinates with the following boundary conditions:

b.c.

$$r=0 \quad V \neq \infty$$

$$r \rightarrow \infty \quad V_2 = 0$$

$$r=R \quad V_1(r=R) = V_2(r=R) = V_0 = k \cos 3\theta$$

$$r=R \quad \frac{\partial V_2}{\partial r} \Big|_{r=R} \quad \frac{\partial V_1}{\partial r} \Big|_{r=R} = -\frac{\sigma}{\epsilon}$$

$$\nabla^2 V(r, \theta, \phi) = 0$$

Using the problem has azimuthal symmetry we have

$$\nabla^2 V(r, \theta) = 0$$

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The general solution of this equation is

1) Inside the sphere  $r < R$

$$V_1(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

apply b.c.  $V_1(r=0, \theta) = \text{finite} \Rightarrow B_l = 0$

$$\Rightarrow V_1(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

2) Outside the sphere

$$V_2(r, \theta) = \sum_{l=0}^{\infty} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos\theta)$$

apply b.c.  $V_2(r \rightarrow \infty, \theta) = 0 \Rightarrow C_l = 0$

$$\Rightarrow V_2(r, \theta) = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta)$$

at  $r=R$   $V_1(r, \theta) = V_2(r, \theta) = k \cos(3\theta)$

we use that  $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

$$(1) \Rightarrow \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = k (4\cos^3\theta - 3\cos\theta)$$

The highest power of  $\cos\theta$  present in the right-hand-side of (1) is 3. Therefore the highest order of Legendre polynomial in the sum of the left-hand side in (1) is 3 or  $A_l = 0$  for  $l \geq 3$

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Write (1) in detail

$$(2) A_0 + A_1 R \cos\theta + A_2 R^2 \left[ \frac{1}{2} (3 \cos^2\theta - 1) \right] + A_3 R^3 \left[ \frac{1}{2} (5 \cos^3\theta - 3 \cos\theta) \right] = \\ = 4k \cos^3\theta - 3k \cos\theta$$

Group the terms on the left-hand-side of (2)  
with the same power of  $\cos\theta$

$$(3) \left( A_0 - \frac{1}{2} A_2 R^2 \right) + \left( A_1 R - \frac{3}{2} A_3 R^3 \right) \cos\theta + \left( \frac{3}{2} A_2 R^2 \right) \cos^2\theta + \\ + \left( \frac{5}{2} A_3 R^3 \right) \cos^3\theta = (-3k) \cos\theta + (4k) \cos^3\theta$$

The coefficients in front of the equal powers of  
 $\cos\theta$  on both sides of (3) should be equal

$$\Rightarrow \left. \begin{array}{l} \left( A_0 - \frac{1}{2} A_2 R^2 \right) = 0 \\ \left( \frac{3}{2} A_2 R^2 \right) = 0 \end{array} \right\} \begin{array}{l} A_0 = 0 \\ A_2 = 0 \end{array}$$

$$\left( \frac{5}{2} A_3 R^3 \right) = 4k \Rightarrow A_3 = \frac{8}{5} \frac{k}{R^3}$$

$$\left( A_1 R - \frac{3}{2} A_3 R^3 \right) = -3k \Rightarrow A_1 = \frac{1}{R} \left( \left( \frac{3}{2} A_3 R^3 \right) - 3k \right)$$

$$A_1 = \frac{3}{5} \frac{k}{R}$$

The solution inside the sphere is

$$V_1(r, \theta) = \frac{k}{5} \left[ 8 \left( \frac{r}{R} \right)^3 P_3(\cos\theta) - 3 \left( \frac{r}{R} \right) P_1(\cos\theta) \right]$$

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Outside the sphere:

$$V_2(r, \theta) = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta)$$

but for  $r=R$   $V_1(r, \theta) = V_2(r, \theta)$

$$\Rightarrow \sum_{l=0}^{\infty} \frac{D_l}{R^{l+1}} P_l(\cos\theta) = \frac{8}{5} k P_3(\cos\theta) - \frac{3}{5} k P_1(\cos\theta)$$

$$\Rightarrow D_l = 0 \text{ for } l=0, 2, 4, 5, \dots$$

$$\frac{D_1}{R^2} = -\frac{3}{5} k \Rightarrow D_1 = -\frac{3kR^2}{5}$$

$$\frac{D_3}{R^4} = \frac{8}{5} k \Rightarrow D_3 = \frac{8}{5} k R^4$$

The solution outside the sphere is

$$V_2(r, \theta) = \frac{8}{5} k \left(\frac{R}{r}\right)^4 P_3(\cos\theta) - \frac{3k}{5} \left(\frac{R}{r}\right)^2 P_1(\cos\theta)$$

To find  $\sigma$  we need to use

$$\left. \frac{\partial V_1}{\partial r} \right|_{r=R} - \left. \frac{\partial V_1}{\partial r} \right|_{r=R} = -\frac{\sigma}{\epsilon}$$

$$\left. \frac{\partial V_1}{\partial r} \right|_{r=R} = \frac{k}{5} \left[ 24 \frac{r^2}{R^3} P_3(\cos\theta) - 3 \frac{1}{R} P_1(\cos\theta) \right]$$

$$\left. \frac{\partial V_1}{\partial r} \right|_{r=R} = \frac{3}{5} k \frac{1}{R} \left[ 8 P_3(\cos\theta) - P_1(\cos\theta) \right]$$

$$\frac{\partial V_2}{\partial r} = \frac{-8k}{5} \frac{4R^4}{r^5} P_3(\cos\theta) - \frac{3k}{5} \frac{2R^2}{r^3} P_1(\cos\theta) =$$

$$= \frac{2k}{5} \left[ -16 \frac{R^4}{r^5} P_3(\cos\theta) + 3 \frac{R^2}{r^3} P_1(\cos\theta) \right]$$

$$\left. \frac{\partial V_2}{\partial r} \right|_{r=R} = \frac{2k}{5} \left[ -16 \frac{1}{R} P_3(\cos\theta) + 3 \frac{1}{R} P_1(\cos\theta) \right]$$

$$\left. \frac{\partial V_2}{\partial r} \right|_{r=R} = \left. \frac{\partial V_1}{\partial r} \right|_{r=R} = \frac{k}{5R} \left[ -32 P_3(\cos\theta) + 6 P_1(\cos\theta) - 24 P_3(\cos\theta) + 3 P_1(\cos\theta) \right] = -\frac{\sigma}{\epsilon}$$

$$-56 P_3(\cos\theta) + 9 P_1(\cos\theta) \right] \frac{k}{5R} = -\frac{\sigma}{\epsilon}$$

$$\sigma = \frac{6k}{5R} \left( 56 P_3(\cos\theta) - 9 P_1(\cos\theta) \right)$$