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Exact results for the Kondo effect in a Luttinger liquid

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The Kondo effect in a Luttinger liquid composed of right-moving, spin-up electrons and left-moving, spin-down electrons is mapped exactly onto the Kondo effect in a Fermi liquid. The transformation generates anisotropy in the exchange coupling, which explains the two most notable features of the Kondo effect in a full Luttinger liquid: the quenching of the impurity moment for ferromagnetic, as well as antiferromagnetic, exchange and a power-law dependence of the Kondo temperature on the exchange coupling. Impurity contributions to the low-energy thermodynamics have the same temperature dependence as in a Fermi liquid.

The development of nanofabrication techniques for thin metallic wires has renewed interest in one-dimensional electron systems. Such systems differ fundamentally from those in three dimensions, where electron-electron interactions can be absorbed into a local Fermi-liquid description of weakly interacting quasiparticles. In the presence of repulsive interactions, quasiparticle excitations are replaced in one dimension by collective density excitations.[1,2] The Luttinger-liquid picture of the low-temperature properties features anomalous power-law decay of various correlation functions with exponents that vary smoothly with the interaction strength.

Recently, considerable attention has been focused on the response of a Luttinger liquid to localized perturbations. The phenomena studied include tunneling through potential barriers[3] and the x-ray edge problem.[4] The exchange interaction between the electron gas and a localized magnetic impurity falls into the same category. In three dimensions, this interaction gives rise to the well-known Kondo effect:[5-7] As the temperature T drops below a characteristic scale T_K, the local moment becomes progressively screened by the conduction sea. At $T=0$ the local degree of freedom is completely quenched, while the low-lying excitations are those of a Fermi liquid. It is natural to ask in what form (if any) the Kondo effect persists in one dimension, where the electrons are strongly interacting.

The Kondo effect in a Luttinger liquid was first considered by Lee and Toner,[8] who used Abelian bosonization to map the problem onto a kink-gas action. Perturbative renormalization-group techniques were then applied to show that the effective exchange coupling scales from weak coupling towards strong coupling, as is the case in three dimensions. However, the dependence of the Kondo temperature T_K on the bare exchange J crosses over from the familiar exponential form in the absence of correlations to a power-law form when the Coulomb interaction is appreciably larger than J. Subsequently, a poor-man's scaling treatment by Furusaki and Nagaosa[9] revealed that, unlike a local Fermi liquid, a Luttinger liquid supports a Kondo effect even if the bare exchange interaction is ferromagnetic. The difference was attributed to the presence of Kondo backward scattering.

The overall picture is incomplete because weak-coupling analysis reveals little about the low-temperature physics. In Ref. 9, a local spin singlet was posited to form at the strong-coupling fixed point. This conjecture, which results in anomalous exponents for the impurity contribution to the specific heat, remains to be verified.

In this paper we show that the main features noted in Refs. 8 and 9 can be recovered from a simpler model which can be treated nonperturbatively. We consider the Kondo effect in a Luttinger liquid that consists of one branch of right-moving, spin-up electrons and one branch of left-moving, spin-down electrons. This model can be mapped *exactly* onto the Kondo effect in a local Fermi liquid, the effect of interactions being simply to renormalize parameters. An anisotropy introduced into the exchange coupling accounts for the J dependence of the Kondo temperature and the existence of a ferromagnetic Kondo effect. The low-temperature regime is characterized by the strong-coupling fixed point for the Kondo effect in a Fermi liquid. While we know of no direct realization of the model, it can be recast as an extension of Emery and Kivelson's orbital Kondo problem[10] to a spinless Luttinger liquid. As such, it may also be related to the problem of a spin-defect in a spin-$\frac{1}{2}$ Heisenberg chain.[11,12]

We shall work within an effective low-energy description in terms of continuum fields.[1] Our starting point is the Hamiltonian $\mathcal{H}=\mathcal{H}_{\rm Lutt}+\mathcal{H}_{\rm imp}+\mathcal{H}_{\rm mag}$, where

$$\mathcal{H}_{\rm Lutt}=-iv_F\sum_{s=\pm} s\int_{-\infty}^{\infty}\psi_s^\dagger(x)\frac{\partial}{\partial x}\psi_s(x)dx + Ua\int_{-\infty}^{\infty}:\psi_+^\dagger(x)\psi_+(x):\ :\psi_-^\dagger(x)\psi_-(x):dx\ ,$$

$$\mathcal{H}_{\rm imp}=\frac{J_z a}{2}\sum_s s\psi_s^\dagger(0)\psi_s(0)\tau^z + \frac{J_\perp a}{2}\sum_{\lambda=x,y}\sum_{s,s'}\psi_s^\dagger(0)\sigma_{s,s'}^\lambda\psi_{s'}(0)\tau^\lambda\ , \qquad (1)$$

$$\mathcal{H}_{\rm mag}=-H\left[\frac{g_e}{2}\sum_s s\int_{-\infty}^{\infty}\psi_s^\dagger(x)\psi_s(x)dx+g_i\tau^z\right]\ .$$

Here $\psi_+(x)$ [$\psi_-(x)$] annihilates a right-moving, spin-up [left-moving, spin-down] electron at position x; $\mathcal{H}_{\rm Lutt}$ describes an unperturbed Luttinger liquid with Fermi velocity v_F and interaction strength U; $\mathcal{H}_{\rm imp}$ represents the exchange interaction between the Luttinger liquid and a local spin-$\frac{1}{2}$ impurity, $\vec{\tau}$; and $\mathscr{H}_{\rm mag}$ introduces the effect of an applied magnetic field H. In these equations, "::" indicates normal ordering, a is a short-distance cutoff, and σ^λ is a Pauli matrix. We allow for anisotropy in the bare exchange coupling (J_z and $J_\perp$), as well as the possibility of the electrons and the impurity having different Landé g factors (g_e and g_i, respectively). Eventually we shall focus on the case where $J_z=J_\perp=J$ and $g_e=g_i=g$. We shall also assume a repulsive Coulomb interaction, $0<U<2\pi v_F/a$. (For $U=0$, the model reduces to that for the standard Kondo effect in a Fermi liquid.)

The Hamiltonian can be bosonized by writing[1,2,13]

$$\psi_\pm(x)=\frac{1}{\sqrt{2\pi a}}\exp\left[-i\sqrt{\pi}\left(\int_{-\infty}^{x}\Pi(x')dx'\mp\phi(x)\right)\right], \quad (2)$$

where $\phi(x)$ and $\Pi(x)$ are real, conjugate Bose fields that obey standard commutation relations, $[\phi(x),\Pi(y)]=i\delta(x-y)$. In terms of these Bose fields, Eqs. (1) become

$$\mathscr{H}_{\rm Lutt}=\frac{v_F}{2}\int_{-\infty}^{\infty}\left[\left(1-\frac{Ua}{2\pi v_F}\right)\Pi(x)^2+\left(1+\frac{Ua}{2\pi v_F}\right)\times\left(\frac{\partial\phi(x)}{\partial x}\right)^2\right]dx ,$$

$$\mathscr{H}_{\rm imp}=-\frac{J_z a}{2\sqrt{\pi}}\Pi(0)\tau^z+\frac{J_\perp}{4\pi}i[e^{i2\sqrt{\pi}\phi(0)}\tau^+ - e^{-i2\sqrt{\pi}\phi(0)}\tau^-] , \quad (3)$$

$$\mathscr{H}_{\rm mag}=-H\left[-\frac{g_e}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\Pi(x)dx+g_i\tau^z\right] ,$$

where $\tau^\pm=\tau^x\pm i\tau^y$. Thus, $\mathscr{H}_{\rm Lutt}$ describes a noninteracting problem, which can be cast into free-boson form via the canonical transformation $\tilde{\Pi}(x)=K^{1/2}\Pi(x)$, $\tilde{\phi}(x)=K^{-1/2}\phi(x)$. Here

$$K=\left[\left(1-\frac{Ua}{2\pi v_F}\right)\bigg/\left(1+\frac{Ua}{2\pi v_F}\right)\right]^{1/2}<1 \quad (4)$$

is a characteristic parameter of the host liquid (analogous to K_ρ and K_σ, which determine the zero-temperature correlation exponents of the full Luttinger liquid). In terms of the new fields the Hamiltonian becomes

$$\mathscr{H}_{\rm Lutt}=\frac{\alpha v_F}{2}\int_{-\infty}^{\infty}\left[\tilde{\Pi}(x)^2+\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)^2\right]dx ,$$

$$\mathscr{H}_{\rm imp}=-\frac{J_z a}{2\sqrt{\pi}}K^{-1/2}\tilde{\Pi}(0)\tau^z+\frac{J_\perp}{4\pi}i\{e^{i2\sqrt{\pi K}\tilde{\phi}(0)}\tau^+ - e^{-i2\sqrt{\pi K}\tilde{\phi}(0)}\tau^-\} , \quad (5)$$

$$\mathscr{H}_{\rm mag}=-H\left[-\frac{g_e}{2\sqrt{\pi}}K^{-1/2}\int_{-\infty}^{\infty}\tilde{\Pi}(x)dx+g_i\tau^z\right] ,$$

with

$$\alpha=\left[1-\left(\frac{Ua}{2\pi v_F}\right)^2\right]^{1/2}<1. \quad (6)$$

While the Coulomb interaction term has been conveniently absorbed into a renormalization of the Fermi velocity, the $\tau^\pm$ terms in the impurity part of Eqs. (5) acquire an exponent that depends explicitly on $K^{1/2}$. In general, such an exponent is difficult to handle, but in this case it can be removed by applying a second canonical transformation. By analogy with Ref. 14 we consider a rotation in τ space: $\tilde{\mathscr{H}}=\hat{U}\mathscr{H}\hat{U}^\dagger$, $\hat{U}=\exp[i\lambda\tilde{\phi}(0)\tau^z]$, where λ is a real number to be determined. The components of $\mathscr{H}$ transform according to[14]

$$\mathscr{H}_{\rm Lutt}\to\mathscr{H}_{\rm Lutt}-\lambda\alpha v_F\tilde{\Pi}(0)\tau^z ,$$
$$\tau^\pm\to e^{\pm i\lambda\tilde{\phi}(0)}\tau^\pm , \quad (7)$$
$$\mathscr{H}_{\rm mag}\to\mathscr{H}_{\rm mag}-H\frac{\lambda g_e}{2\sqrt{\pi}}K^{-1/2}\tau^z .$$

Hence, by selecting $\lambda=2\sqrt{\pi}(1-K^{1/2})$ we can cancel the factor of $K^{1/2}$ in the exponentials preceding $\tau^\pm$. The new terms that are generated in the course of this rotation simply renormalize the original parameters of the model. Finally, one can get back to a fermionic representation by defining a new set of fermion fields,

$$\tilde{\psi}_\pm(x)=\frac{1}{\sqrt{2\pi a}}\exp\left[-i\sqrt{\pi}\left(\int_{-\infty}^{x}\tilde{\Pi}(x')dx'\mp\tilde{\phi}(x)\right)\right]. \quad (8)$$

After all these manipulations, we arrive at the following Hamiltonian:

$$\tilde{\mathscr{H}}=-i\tilde{v}_F\sum_{s=\pm}s\int_{-\infty}^{\infty}\tilde{\psi}_s^\dagger(x)\frac{\partial}{\partial x}\tilde{\psi}_s(x)dx + \frac{\tilde{J}_z a}{2}\sum_s s\tilde{\psi}_s^\dagger(0)\tilde{\psi}_s(0)\tau^z + \frac{J_\perp a}{2}\sum_{\lambda=x,y}\sum_{s,s'}\tilde{\psi}_s^\dagger(0)\sigma^\lambda_{s,s'}\tilde{\psi}_{s'}(0)\tau^\lambda - \tilde{H}\left[\frac{g_e}{2}\sum_s s\int_{-\infty}^{\infty}\tilde{\psi}_s^\dagger(x)\tilde{\psi}_s(x)dx+\tilde{g}_i\tau^z\right] , \quad (9)$$

where

$$\tilde{v}_F=\alpha v_F,$$
$$\tilde{J}_z=K^{-1/2}J_z+\frac{4\pi v_F}{a}\alpha(1-K^{1/2}),$$
$$\tilde{H}=K^{-1/2}H, \quad (10)$$
$$\tilde{g}_i=g_e+K^{1/2}(g_i-g_e).$$

Equations (9) and (10) constitute the main technical result of this paper. They provide an exact mapping of the model in Eqs. (1) onto the Kondo Hamiltonian for a noninteracting conduction band. Within this mapping, the role of the Coulomb interaction U is merely to renormalize the Fermi velocity (i.e., the density of states), the applied magnetic field, the impurity Landé g factor, and the z component of the Kondo coupling. Since $J_\perp$ is unaffected, an originally isotropic coupling, $J_z=J_\perp=J$, necessarily becomes anisotropic. For the case $g_i=g_e=g$, the impurity g factor is also unaffected by the mapping.

We can use this result to show that the principal features of the Kondo effect in a full Luttinger liquid[8,9] also appear in our model, where they stem directly from the exchange anisotropy generated during the mapping. For simplicity we focus on the limit $Ua\ll 2\pi v_F$, in which $\tilde{v}_F\simeq v_F$ and $\tilde{J}_z\simeq J_z+U$. However, the qualitative picture does not depend on this assumption.

Based on the perturbative renormalization-group approach for the conventional Kondo problem,[15] the scaling equations for the coupling constants $\tilde{J}_z$ and $J_\perp$ are

$$\frac{d\tilde{J}_z}{d\ln(E_0/E)}=\frac{J_\perp^2 a}{2\pi v_F}\ , \quad \frac{dJ_\perp}{d\ln(E_0/E)}=\frac{\tilde{J}_z J_\perp a}{2\pi v_F}\ , \tag{11}$$

where E is a renormalized cutoff and E_0 is the bare bandwidth. The system flows to strong coupling, and undergoes a Kondo effect, for bare couplings satisfying $\tilde{J}_z>-|J_\perp|$, and flows to weak coupling otherwise.[15]

For a repulsive Coulomb interaction and an isotropic bare Kondo coupling, $\tilde{J}_z$ always exceeds $-|J_\perp|$, so there is a Kondo effect *regardless of whether the exchange coupling is ferromagnetic or antiferromagnetic*.[9] For anisotropic bare couplings, a Kondo effect will occur provided $J_z+U>-|J_\perp|$. A sufficiently large U, therefore, drives the system towards a Kondo effect, even if the bare couplings are highly anisotropic and ferromagnetic, i.e., $-J_z/|J_\perp|\gg 1$.

The Kondo temperature T_K can be associated with the cutoff E at which the coupling constants become of order the bare bandwidth, $E_0=\pi v_F/a$. For an isotropic bare exchange (antiferromagnetic or ferromagnetic) satisfying $J+U>|J|$, integration of Eqs. (11) yields

$$T_K=E_0\exp\left[-\frac{2E_0}{\xi}\sinh^{-1}\left(\frac{\xi}{|J|}\right)\right]\ , \tag{12}$$

where $\xi=\sqrt{U^2+2UJ}$. For $U\ll J$, Eq. (12) reduces to the familiar expression $T_K=E_0\exp(-2E_0/J)$, while for $U\gg|J|$, the power-law dependence

$$T_K=E_0\left(\frac{|J|}{2U}\right)^{2E_0/U} \tag{13}$$

is obtained.[8,9] Note that even in the latter case T_K remains exponential in U, which in this limit is the dominant Kondo coupling entering $\tilde{\mathcal{H}}$. The power-law dependence on J is to be understood as a weaker dependence on a minor coupling constant. Between the two extremes, T_K evolves continuously as a function of U, as do all impurity-related properties.

For an isotropic, ferromagnetic exchange such that $U+J<|J|$, the above expressions for T_K no longer apply. In particular, when $U+J<0$, the Kondo couplings initially diminish before scaling to strong coupling. Accordingly, a nonmonotonic temperature dependence of physical quantities is to be expected.[9]

Our mapping has clear advantages over previous treatments when it comes to the strong-coupling regime, $T<T_K$, since the thermodynamics must be those of a Fermi liquid. Specifically, all impurity contributions to thermodynamic quantities reduce to functions of the single energy scale T_K.[6,7] For instance, the susceptibility $\chi_{\rm imp}$ and the specific heat coefficient $C_{\rm imp}/T$ are both inversely proportional to T_K in the limit $T\to 0$.

Another quantity of interest is the Wilson ratio $R_W=\lim_{T\to 0}(\chi_{\rm imp}C_{\rm bulk})/(\chi_{\rm bulk}C_{\rm imp})$. For the standard Kondo effect with $g_i=g_e=g$, this takes the universal value[6] $R_W=2$. Mapping the Luttinger liquid onto a noninteracting electron gas increases the effective magnetic field by a factor of $K^{-1/2}$ [see Eqs. (10)]. In the case of equal g factors, this enhances both $\chi_{\rm imp}$ and $\chi_{\rm bulk}$ by a factor of K^{-1}. The overall effect, therefore, is that the Wilson ratio remains equal to 2.

While thermodynamic quantities can, in principle, be obtained to arbitrary accuracy within our approach (e.g., via the Bethe ansatz[7] or the numerical renormalization-group method[6]), transport properties are hard to extract. The latter must be computed from correlation functions of the *physical* conduction electrons, rather than the noninteracting fermions entering Eqs. (9). When an impurity-free Luttinger liquid is mapped onto a free-fermion problem, the nontrivial relation between real and fictitious fermions is precisely the origin of the anomalous power laws[1,2] which distinguish the system from a true Fermi liquid. It is also the key to the exotic tunneling phenomena in Luttinger liquids.[3]

The extension of the present mapping to the Kondo effect in a full (four-branch) Luttinger liquid is complicated by the need for *two* sets of Bose fields to represent the conduction electrons. This leads to a technical difficulty in finding a suitable rotation in τ space that maps the impurity Hamiltonian $\mathcal{H}_{\rm imp}$ into any form that can be expressed in terms of otherwise noninteracting fermions, let alone the particular form of the Kondo Hamiltonian.

It is therefore worthwhile to discuss the relevance of our results to the full Luttinger liquid. We begin by emphasizing that the various weak-coupling regimes (both antiferromagnetic and ferromagnetic) described above coincide precisely with those found for the isotropic Kondo effect in a full Luttinger liquid.[9] We also note that an alternative two-branch model, with both spin-up and spin-down electrons moving in the *same* direction, can also be mapped onto the Kondo Hamiltonian for a noninteracting conduction band. In this case, though, only the Fermi velocity is renormalized by the mapping, so both the criterion for the occurrence of the Kondo effect and the J dependence of the Kondo temperature are the same as in three dimensions. That the two types of spin move in opposite directions is thus seen to be a crucial feature of the model studied in this paper. It also confirms the observation in Ref. 9 that the presence of Kondo backward-scattering terms accounts for the fundamental difference between a Kondo effect in a Luttinger liquid and that in a Fermi liquid. In the weak-coupling limit, then, we find that the two-branch and four-branch models share qualitatively the same Kondo physics.

The low-temperature physics may be a different matter. Furusaki and Nagaosa[9] conjectured that the full Luttinger liquid flows to an infinite-coupling fixed point, at which the impurity forms a spin singlet with a physical electron at the impurity site, plus — in the case of ferromagnetic exchange — one electron at each site on either side of the impurity. The system is thereby broken into two disconnected, impurity-free Luttinger liquids. Those authors established, via a $1/J$ expansion, that this state is locally stable, and went on to predict an anomalous exponent for the temperature-dependence of $C_{\rm imp}$. The mapping of the two-branch problem onto the standard Kondo model guarantees that the system flows to $J=\infty$, and leads to a ground-state spin singlet of the transformed fermions. However, the representation of this state using physical electrons is more complicated. Moreover, as stated above, $C_{\rm imp}$ must have the regular linear temperature dependence. It remains to be seen whether these discrepancies originate from the ground-state ansatz made in Ref. 9 or from an essential difference between the two- and four-branch models.

In conclusion, we have shown that the Kondo effect in a Luttinger liquid in which the up and down spins move in opposite directions can be understood via an exact mapping onto the Kondo effect in a Fermi liquid. The qualitative differences between the two cases — the irrelevance of the sign of the exchange coupling J for the occurrence of Kondo quenching, and the power-law dependence of T_K on J when the exchange is much weaker than the interaction within the conduction band (i.e., $|J| \ll U$) — both result from an anisotropy that is inevitably generated within the mapping. For the strong-coupling regime, no qualitative changes are seen in thermodynamic quantities.

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