PHY 2060 Fall 2006

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Sample Exam Questions on Special Relativity

The problems below (adapted from various sources) range from fairly easy (number 1) to hard (number 5).

1. If an airline pilot flies 80 hours per month (in her rest frame) at 200 m/s for 25 years, how much younger will she be than her twin brother (who works in the airport control tower) when she retires?

This is an example of time dilation.

The brother sees his sister's internal clock run slow during the time she is flying. If the pilot's internal clock (i.e., her age) advances by Δt_0, then her brother's internal clock advances by

$$\Delta t = \gamma \Delta t_0$$

so the difference in their ages is

$$\Delta t - \Delta t_0 = (\gamma - 1)\Delta t_0 \simeq \frac{1}{2}\left(\frac{v}{c}\right)^2 \Delta t_0$$

since $\gamma \simeq 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$ for $v \ll c$.

The total difference in their ages over 25 years is

$$\Delta t - \Delta t_0 \approx \frac{1}{2}\left(\frac{200\,\text{m/s}}{3.0\times10^8\,\text{m/s}}\right)^2 \times 80\,\text{hours/month} \times \left(\frac{3600\,\text{s}}{1\,\text{hour}}\right) \times \left(\frac{12\,\text{months}}{1\,\text{year}}\right) \times 25\,\text{years}$$

$$\simeq 19\,\mu\text{s}$$

2. A train of proper length L_0 approaches a station at a constant speed u in the station's rest frame. An observer at the station observes that the front of the train passes one end of the platform (call this event A) at the same moment as the rear of the train passes the other end of the platform (event B). In the rest frame of the train, what is the time interval between the events, and which event occurs first?

In the station frame, the train has length

$$L = L_0/\gamma$$

where

$$\gamma = [1-(u/c)^2]^{-\frac{1}{2}}.$$

Taking the train's motion to be along the $+x$ axis,

$$\Delta x = x_A - x_B = L_0/\gamma,$$

$$\Delta t = t_A - t_B = 0.$$

In the train's rest frame

$$\Delta x' = \gamma(\Delta x - u\Delta t)$$

$$= L_0 \quad \text{as expected}$$

$$\Delta t' = t_A' - t_B' = \gamma(\Delta t - u\Delta x/c^2)$$

$$= -L_0 u/c^2.$$

Thus, the time interval between the events in the train's frame is $L_0 u/c^2$, with event A occurring first.

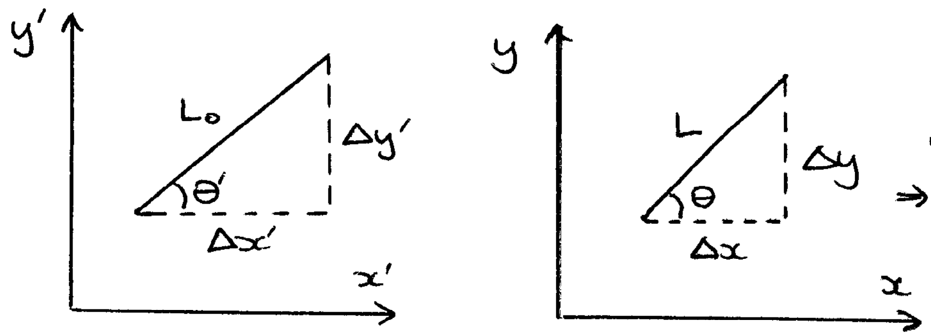
3. A rod of proper length L_0 is at rest in a reference frame S'. It lies in the $x'y'$ plane making an angle of $\sin^{-1}(3/5)$ with the positive x' axis. S' moves with constant speed u parallel to the positive x axis of another frame S.

(a) What is the value of u if, as measured in S, the rod is oriented at 45° to the x axis?

(b) What is the length of the rod as measured in S under these conditions?

$\sin\theta' = \frac{3}{5}$

$\Rightarrow \tan\theta' = \frac{3}{4}$



$\theta = 45°$

$\Rightarrow \tan\theta = 1$

(a) In S, the rod makes an angle θ satisfying

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'/\gamma}$$

$\leftarrow$ no transverse contraction

$\leftarrow$ length contraction

$$= \gamma \tan\theta'$$

$$\Rightarrow \quad \gamma = \frac{\tan\theta}{\tan\theta'} = \frac{1}{4/3}$$

$$= \frac{4}{3}$$

$$v = c\sqrt{1-\gamma^{-2}} = c\sqrt{1-\tfrac{9}{16}}$$

$$= \sqrt{\tfrac{7}{16}}\, c$$

(b) The length in S is

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{\left(\frac{\Delta x'}{\gamma}\right)^2 + (\Delta y')^2}$$

$$= L_0\sqrt{\left(\tfrac{3}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2}$$

$$= \frac{3\sqrt{2}}{5} L_0$$

4. Boris, standing at the rear end of a railroad train, shoots an arrow toward the front of the train. The speed of the arrow as measured by Boris is $c/5$. The length of the train as measured by Boris is 150 m. Natasha, standing on the station platform, observes all of this as the train passes by her with a velocity of $3c/5$. What values does Natasha measure for the following quantities:

(a) The length of the train?

(b) The speed of the arrow?

(c) The length of the time the arrow is in the air?

(d) The distance that the arrow travels?

(a) Due to length contraction,

$$L = \frac{L_0}{\gamma}$$

where

$$\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-\frac{1}{2}} = \left[1 - \left(\frac{3}{5}\right)^2\right]^{-\frac{1}{2}} = \frac{5}{4}$$

$$\Rightarrow \quad L = \frac{150\,\mathrm{m}}{5/4} = 120\,\mathrm{m}$$

(b) Velocity addition:

$$V_x = \frac{V_x' + U}{1 + UV_x'/c^2} = \frac{c/5 + 3c/5}{1 + (3/5)(1/5)}$$

$$= \frac{5c}{7} = 2.14 \times 10^8\,\mathrm{m/s}$$

(c) Suppose the arrow is launched at time $t = 0$ in Natasha's frame.

Then its position is

$$x_{arrow} = x_0 + V_x t$$

while that of the front of the train is

$$x_{front} = x_0 + L + Ut$$

The arrow reaches the front when

$$x_{arrow} = x_{front}$$

$$t = \frac{L}{V_x - U} = \frac{120\,\mathrm{m}}{\frac{5}{7}c - \frac{3c}{5}}$$

$$= 3.5\,\mathrm{ps}$$

(d) distance

$$\Delta x = V_x t = (2.14 \times 10^8\,\mathrm{m/s})(3.5 \times 10^{-6}\,\mathrm{s})$$

$$= 750\,\mathrm{m}$$

5. Suppose that in frame S, a certain object has speed $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Show that its speed v' in a frame S' moving at constant speed u along the $+x$ axis of S satisfies

$$(v'/c)^2 = 1 - \frac{1-(u/c)^2}{(1-uv_x/c^2)^2}[1-(v/c)^2].$$

Assuming that $u < c$, show that ...

(a) if the object's speed in S satisfies $v < c$, then its speed in S' necessarily satisfies $v' < c$;

(b) if the object has speed $v = c$ in S, then its speed in S' is necessarily $v' = c$.

Using the Lorentz velocity transformation,

$$v'^2 = v_x'^2 + v_y'^2 + v_z'^2 = \frac{(v_x - u)^2 + \gamma^{-2}(v_y^2 + v_z^2)}{(1 - uv_x/c^2)^2}$$

The denominator

$$\begin{aligned}
&= v_x^2 - 2uv_x + u^2 + (1 - u^2/c^2)(v_y^2 + v_z^2) \\
&= \frac{u^2}{c^2}v_x^2 - 2uv_x + u^2 + (1 - u^2/c^2)(v_x^2 + v_y^2 + v_z^2) \\
&= \left(\frac{u^2 v_x^2}{c^2} - 2uv_x + c^2\right) - (c^2 - u^2) + \left(1 - \frac{u^2}{c^2}\right)v^2 \\
&= (c - uv_x/c)^2 - c^2\left(1 - \frac{u^2}{c^2}\right) + (1 - u^2/c^2)v^2 \\
&= c^2\left[(1 - uv_x/c^2)^2 - (1 - u^2/c^2)(1 - v^2/c^2)\right]
\end{aligned}$$

$$\Rightarrow \quad \left(\frac{v'}{c}\right)^2 = 1 - \frac{1-(u/c)^2}{(1-uv_x/c^2)^2}\left[1-(v/c)^2\right] \quad \text{as claimed}$$

(a)

$$1 - \left(\frac{v'}{c}\right)^2 = \frac{1-(u/c)^2}{(1-uv_x/c^2)^2}\left[1-(v/c)^2\right] \qquad (1)$$

For $u < c$ and $v < c$, the right-hand side must be positive

$$\Rightarrow \quad 1 - (v'/c)^2 > 0$$

$$(v'/c)^2 < 0$$

$$v' < c.$$

(b) For $v = c$, the right-hand side of (1) must be zero.

$$\Rightarrow \quad 1 - (v'/c)^2 = 0$$

$$v' = c.$$