PHY 2060 Fall 2006 – Exam 1 Solution

1. The position of a particle moving along the x axis is given by $x = a + bt - ct^2$, where a, b, and c are positive constants.

[3 points] (a) Find the velocity as a function of time.

[3 points] (b) What is the average velocity over the time interval $0 \le t \le 2$?

[4 points] (c) At what time is the particle at its rightmost point, and what is its position at that time?

(a) Position $\quad x(t) = a + bt - ct^2$

$\Rightarrow$ velocity

$$v(t) = \frac{dx}{dt} = b - 2ct$$

(b) Average velocity

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(2) - x(0)}{2 - 0} = \frac{(a + 2b - 4c) - (a)}{2} = b - 2c$$

(c) Extrema of x occur at times t where

$$v(t) = 0$$

Here, that means

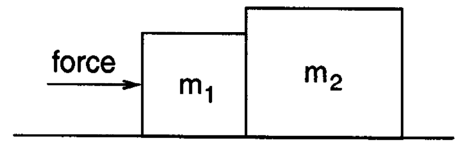
$$t = \frac{b}{2c}$$

at which time

$$x = a + b\left(\frac{b}{2c}\right) - c\left(\frac{b}{2c}\right)^2 = a + \frac{b^2}{4c}$$

Since $v < 0$ for $t > \frac{b}{2c}$, this is clearly the maximum of x, or the rightmost position as the x axis is conventionally drawn.

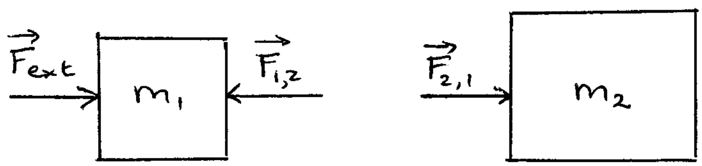
2. Two blocks, of mass m_1 and m_2, lie in contact on a horizontal, frictionless table. A horizontal force applied to the leftmost block, which is m_1, gives both blocks a common acceleration a to the right.



[5 points] (a) What is the magnitude of the applied external force?

[5 points] (b) What is the magnitude of the contact force between the two blocks?

Free-body diagrams, showing horizontal forces only:



(a) Since $\vec{F}_{2,1} = -\vec{F}_{1,2}$ by Newton's 3rd Law, we can treat the two blocks as a single body of mass $m_1 + m_2$ for the purposes of Newton's 2nd Law:

$$\vec{F}_{ext} = (m_1 + m_2)\vec{a}$$

or

$$|\vec{F}_{ext}| = (m_1 + m_2)|\vec{a}| = (m_1 + m_2)\,a$$

(b) Applying Newton's 2nd Law to m_2,

$$\vec{F}_{2,1} = m_2 \vec{a},$$

implying that the contact force has magnitude

$$|\vec{F}_{2,1}| = |\vec{F}_{1,2}| = m_2\,a$$

3. At the moment you pass over Gainesville at the start of a journey to Pluto (5.0 billion km away, as measured from the Earth), your spaceship's dashboard clock reads 1:00 p.m. If you travel at a constant speed of 1.0 billion km/h, what time will the clock read when you reach Pluto?

Method 1: In the rest frame of the spaceship, the Earth-Pluto distance is Lorentz-contracted from L_o to L_o/γ.

Thus, the time for the journey as measured on the spaceship is

$$\Delta t_s = \frac{L_o/\gamma}{v}$$

where $L_o = 5.0 \times 10^9$ km

$v = 1.0 \times 10^9$ km/h

$$\gamma = \left[1-\left(\frac{v}{c}\right)^2\right]^{-\frac{1}{2}} = \left[1-\left(\frac{1.0\times10^9 \text{ km/h}}{3.0\times10^5 \text{ km/s}} \times \frac{1\text{ h}}{3600\text{ s}}\right)^2\right]^{-\frac{1}{2}}$$

$= 2.65$

$$\Rightarrow \qquad \Delta t_s \simeq 1.89\text{ h} \quad \text{or} \quad 1\text{ h } 53\text{ min}$$

Method 2: In the rest frame of the Earth, the time for the journey is

$$\Delta t_e = \frac{L_o}{v} = 5.0\text{ h}$$

Due to time dilation, an observer on Earth finds the spaceship's clock to run slow, yielding an elapsed time

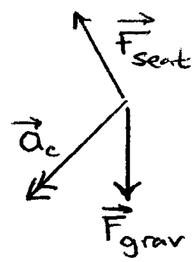
$$\Delta t_s = \frac{\Delta t_e}{\gamma} \simeq 1.89\text{ h}$$ (just as for Method 1)

Observers in both the spaceship's rest frame and the Earth's rest frame agree that the spaceship clock read 1:00 p.m. at the start of the journey and that it reads 2:53 p.m. at the end.

4. A 10-m-diameter Ferris wheel rotates in a vertical plane (so that its rotation axis is horizontal). Each seat can provide at most 400 N of support to its occupant before the seat breaks. Suppose that Jack (mass 35 kg) and Jill (mass 30 kg) are the only passengers, and the speed of the Ferris wheel is gradually increased from zero. At what speed, measured in revolutions per minute, will the first seat break?

   Clarification: Jack and Jill occupy different seats on the Ferris wheel.

The Ferris wheel's speed is increased gradually $\Rightarrow$ can approximate as uniform circular motion, in which each passenger satisfies



$$\sum_i \vec{F}_i = m\vec{a}_c$$
$$\vec{F}_{seat} + \vec{F}_{grav} = -\frac{mv^2}{r}\hat{r}$$
$$\vec{F}_{seat} = -\frac{mv^2}{r}\hat{r} - \vec{F}_{grav}$$

The force exerted by the seat is greatest in magnitude at the lowest point of the rotation.

Also, $|\vec{F}_{seat}| \propto m$, so the more massive passenger (Jack) will require the larger support force.

At the lowest point

$$|\vec{F}_{seat}| = m\left(g + \frac{v^2}{r}\right)$$
$$v = \sqrt{\left(\frac{|\vec{F}_{seat}|}{m} - g\right)r}$$

Angular velocity $\quad \omega = \frac{v}{r} = \sqrt{\left(\frac{|\vec{F}_{seat}|}{m} - g\right)/r}$

Inserting $|\vec{F}_{seat}| = 400\,\text{N}$, $m = 35\,\text{kg}$ and $r = 5.0\,\text{m}$,

$$\Rightarrow \quad \omega = 0.53\ \frac{\text{rad}}{\text{s}} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} \times \frac{60\,\text{s}}{1\,\text{min}}$$
$$= 5.1\ \text{rev/min}$$

5. A ball is thrown straight upwards, and reaches a height h above its release point. The same ball is now thrown with the same initial velocity as before, but this time at an angle of 30° above the horizontal. In both cases, air resistance can be neglected. What is the ball's maximum height above its release point on the second throw?

Let us define the y axis to point vertically upward. $\uparrow y$

On the first throw, we can use the constant-acceleration equation

$$V_y^2 = V_{oy}^2 - 2g(y-y_o) \qquad (1)$$

to find $V_o \equiv V_{oy}$: at the top of the motion

$$0^2 = V_o^2 - 2gh$$

$$V_o^2 = 2gh$$

On the second throw

$$V_{oy} = V_o \sin 30^\circ = \frac{V_o}{2}$$

Using (1) again, at the top of the trajectory

$$0 = (V_o/2)^2 - 2g(y-y_o)$$

$$y - y_o = \frac{V_o^2}{8g}$$

$$= \frac{h}{4}$$

6. At the Solar-Neighborhood Games, competitor A (from Alpha Centauri) approaches the finishing line of the 10,000-km sprint at $0.30\,c$, as measured by an official S (from Sirius) standing beside the track. A is overtaken by entrant B (from Barnard's Star), whose speed as seen by A is $0.60\,c$. However, B doesn't win the race, because he/she/it is passed by runner C (from Chara) traveling at $0.80\,c$, as measured by S.

[5 points] (a) How fast is B traveling, as measured by S?

[5 points] (b) How fast is C traveling, as measured by B?

Give your answers as multiples of c, not in m/s.

Let $\beta_{PQ} = \dfrac{\text{velocity of P in rest frame of Q}}{\text{speed of light}}$

(a) Using the Lorentz velocity transformation with $\frac{v}{c} = \beta_{AS}$

$$\beta_{BS} = \frac{\beta_{BA} + \beta_{AS}}{1 + \beta_{BA}\,\beta_{AS}}$$

$$= \frac{0.60 + 0.30}{1 + (0.60)(0.30)}$$

$$= 0.763$$

$$\simeq 0.76 \qquad \left(\text{As expected, } \beta_{AS} < \beta_{BS} < \beta_{BA} + \beta_{AS}\right)$$

(b) Similarly,

$$\beta_{CB} = \frac{\beta_{CS} - \beta_{BS}}{1 - \beta_{CS}\,\beta_{BS}}$$

$$= \frac{0.80 - 0.763}{1 - (0.80)(0.763)}$$

$$= 0.095 \qquad \left(\text{As expected, } 0 < \beta_{CS} - \beta_{BS} < \beta_{CB}\right)$$

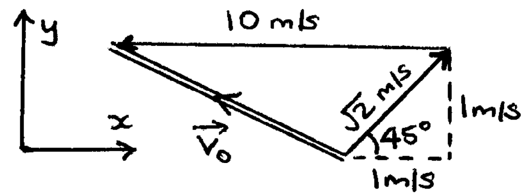
7. On a visit to a mall, a shopper standing on an escalator traveling at $\sqrt{2}$ m/s at an angle of 45° above horizontal throws an empty water bottle horizontally relative to himself at a speed of 10 m/s, back in the direction of the foot of the escalator. The bottle is 4 m above the floor of the mall at the moment it is released.

[5 points] (a) How long does the bottle remain in flight before hitting the floor?

[5 points] (b) How fast is the bottle moving when it lands?

Neglect air resistance and assume that the bottle hits no other object before reaching the floor.

The bottle's initial velocity relative to the floor is the vector sum of its velocity relative to the escalator and the velocity of the escalator relative to the floor.



$$V_{0x} = (\sqrt{2}\cos 45° - 10)\ \text{m/s} = -9\ \text{m/s}$$
$$V_{0y} = \sqrt{2}\sin 45°\ \text{m/s} = +1\ \text{m/s}$$

(a) The time of flight is determined by the vertical motion:

$$y = y_0 + V_{0y}t - \tfrac{1}{2}gt^2$$

Look for time $t = T$ such that $y(T) = 0$:

$$T = \frac{-V_{0y} \pm \sqrt{V_{0y}^2 + 2gy_0}}{-g}$$
$$= \frac{-(1\,\text{m/s}) \pm \sqrt{(1\,\text{m/s})^2 + 2(10\,\text{m/s}^2)(4\,\text{m})}}{-10\,\text{m/s}^2}$$
$$= -0.8\,\text{s or } 1.0\,\text{s}.$$

The bottle remains in the air for 1.0 s after its release.

(b) At time $t = T = 1.0$ s,

$$V_x = V_{0x} = -9\,\text{m/s}$$
$$V_y = V_{0y} - gT = -9\,\text{m/s}$$

$\Rightarrow$ Speed $$V = \sqrt{V_x^2 + V_y^2}$$
$$\simeq 12.7\,\text{m/s}$$

8. Overwhelmed by his Christmas Eve duties, Santa has traded in his sled for a spaceship of proper length 60 m. This spaceship is moving on a straight course past Earth at a constant speed $u = 3c/5$ when Rudolph's nose gives out a flash of red light at the front of the spaceship, and—at the same time, as measured in the spaceship's rest frame—the engines emit a flash of yellow light at the rear of the spaceship. For the purposes of this problem, treat the Earth as an inertial reference frame.

[3 points] (a) What is the spatial distance between the two flashes, as measured in the Earth's frame?

[3 points] (b) Is the answer to (a) the same as the length of the spaceship as measured in the Earth's frame? If not, what is that length?

[4 points] (c) What is the time interval between the light flashes, as measured in the Earth's frame? Which light flash occurs first: the red or the yellow?

In frame S′ (the spaceship),

$$\Delta x' \equiv x'_{red} - x'_{yellow} = 60\,\text{m}$$
$$\Delta t' \equiv t'_{red} - t'_{yellow} = 0$$

(a) In frame S (the Earth)

$$\Delta x = \gamma(\Delta x' + u\Delta t')$$

where $u = \frac{3c}{5}$, $\gamma = [1-(\frac{u}{c})^2]^{-\frac{1}{2}} = \frac{5}{4}.$

$$\Rightarrow \quad \Delta x = \frac{5}{4}\cdot(60\,\text{m} + 0) = 75\,\text{m}.$$

(b) The answer to (a) — obtained for $\Delta t \neq 0$ — is not the length measured in S, which would be

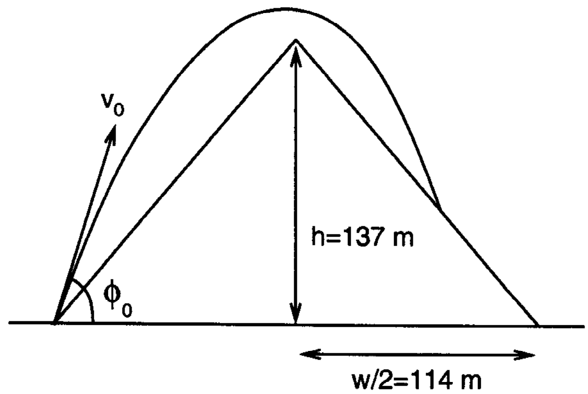
$$L = \frac{L_0}{\gamma} = \frac{4}{5}\times(60\,\text{m}) = 48\,\text{m}.$$

(c) In frame S,

$$\Delta t = \gamma(\Delta t' + u\Delta x'/c^2) = \frac{5}{4}\left(0 + \frac{3}{5}\,\frac{60\,\text{m}}{3\times10^8\,\text{m/s}}\right) = 1.5\times10^{-7}\,\text{s}.$$

Since $\Delta t = t_{red} - t_{yellow} > 0$, $t_{red} > t_{yellow}$, i.e., the yellow flash occurs first in frame S.

9. On a trip to Giza, you can't resist using a slingshot to fire a pebble towards the top of the Great Pyramid. You launch the pebble at 75° above the horizontal at an initial speed of 60 m/s from a point at ground level, right at the foot of the pyramid (see figure). At what height above the ground does the pebble first make contact with the opposite face of the pyramid? Ignore air resistance.



Take the coordinate origin to be the launch point. Then the projectile's path is

$$y = x\tan\phi_0 - \frac{g}{2}\left(\frac{x}{v_0\cos\phi_0}\right)^2 \qquad (1)$$

The equation for the opposite face of the pyramid is

$$y = (w-x)\tan\theta \qquad \text{where } \tan\theta = \frac{h}{w/2} = \frac{2h}{w}$$
$$= 2h - x\tan\theta \qquad (2)$$

The projectile lands at the intersection of lines (1) and (2):

$$x\tan\phi_0 - \frac{g}{2}\left(\frac{x}{v_0\cos\phi_0}\right)^2 = 2h - x\tan\theta$$

$$x = \frac{v_0^2\cos^2\phi_0}{g}\left[(\tan\theta+\tan\phi_0) \pm \sqrt{(\tan\theta+\tan\phi_0)^2 - \frac{4gh}{v_0^2\cos^2\phi_0}}\right]$$

We need the '+' solution. The '-' solution gives the time the projectile cuts the plane of the face in the air to the left of the pyramid's peak

$$\approx 150\text{m}$$

$$y = 2h - x\tan\theta$$
$$\simeq 94\text{m}$$

10. A newly discovered asteroid is found to be on course for a catastrophic collision with Earth. An old Saturn V rocket is refurbished and outfitted with a nuclear warhead designed to divert the asteroid's course. At the moment the rocket is launched, the asteroid's approach speed is 15 km/s, its speed is increasing at a constant rate of $0.05\,\mathrm{m/s^2}$, and at this rate the asteroid will make impact with the Earth in exactly 10 days. If the rocket has a constant acceleration of $5\,\mathrm{m/s^2}$ for the first 20 min of its flight, after which time it maintains a constant velocity, how far from Earth will the rocket meet the asteroid?

For the purposes of this question, assume that the rocket and the asteroid travel along the same straight line, and that all effects of gravity have been taken account in the accelerations given above. Relativistic effects are negligible.

The coordinate of the asteroid can be written

$$x_A = x_{A0} - V_{A0}t - \tfrac{1}{2}a_A t^2$$

where $V_{A0} = 1.5\times10^4\,\mathrm{m/s}$, $a_A = 0.05\,\mathrm{m/s^2}$, and x_{A0} satisfies

$$0 = x_{A0} - V_{0A}T - \tfrac{1}{2}a_A T^2$$

with $T = 10\,\text{days} \times \frac{24\,\text{h}}{1\,\text{day}} \times \frac{3600\,\text{s}}{1\,\text{h}} = 8.64\times10^5\,\text{s}$

$$\Rightarrow \qquad x_{A0} = 3.16\times10^{10}\,\text{m}$$

After time $t = \tau = 20\,\text{min} \times \frac{60\,\text{s}}{1\,\text{min}} = 1200\,\text{s}$, the coordinate of the rocket can be written

$$\begin{aligned} x_R &= \tfrac{1}{2}a_R\tau^2 + a_R\tau(t-\tau) \\ &= a_R\tau t - \tfrac{1}{2}a_R\tau^2 \end{aligned}$$

where $a_R = 5\,\mathrm{m/s^2}$.

The rocket meets the asteroid at time t where

$$x_A(t) = x_R(t)$$

$$\tfrac{1}{2}a_A t^2 + (V_{A0} + a_R\tau)t - (x_{A0} + \tfrac{1}{2}a_R\tau^2) = 0$$

$$\begin{aligned} t &= \frac{-(V_{A0}+a_R\tau) + \sqrt{(V_{A0}+a_R\tau)^2 + 2a_A(x_{A0}+\tfrac{1}{2}a_R\tau^2)}}{a_A} \\ &= 7.80\times10^5\,\text{s} \approx 9.0\,\text{days} \end{aligned}$$

At that time,

$$\begin{aligned} x &= a_R\tau(t - \tfrac{1}{2}\tau) \\ &= 4.68\times10^9\,\text{m} \end{aligned}$$