PHY 2060 Fall 2006 – Exam 3

1. (a) A person pulls a block across a rough horizontal surface at a *constant speed* by applying a force F. The arrows in the diagram below correctly indicate the directions, but not necessarily the magnitudes of the various forces on the block. Which of the following relations among the force magnitudes W, k, N, and F *must be true*?

[max. 5 pts]

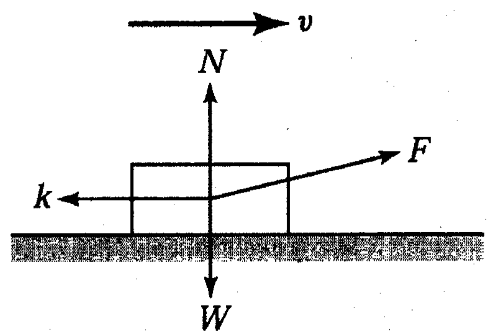
(1) ___ $F = k$ and $N = W$

(2) ___ $F = k$ and $N > W$

(3) ___ $F > k$ and $N = W$

(4) ___ $F < k$ and $N = W$

(5) ___ None of the above



Let the force be at angle θ to the horizontal.

The block is in equilibrium, so

$$\Sigma F_x = 0 \Rightarrow F\cos\theta - k = 0$$

$$F = k\sec\theta$$

$$> k$$

$$\Sigma F_y = 0 \Rightarrow F\sin\theta + N - W = 0$$

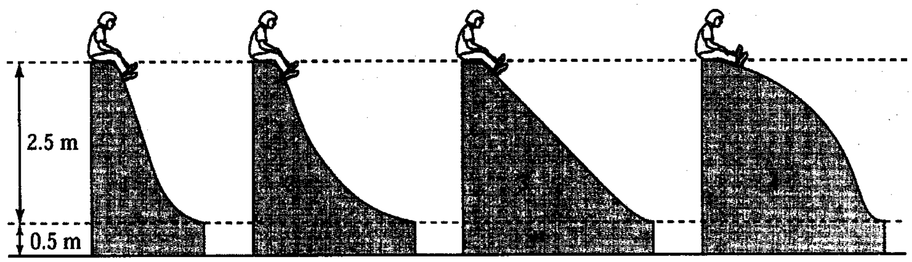
$$N = W - F\sin\theta$$

$$< W$$

Since $F > k$ and $N < W$, the correct answer is (5)

[max. 5 pts]

(b) A young girl wishes to select one of the *frictionless* playground slides illustrated below to give her the greatest possible speed when she reaches the bottom of the slide. Which of the slides illustrated in the diagram should she choose?



(1) ___ Slide 1

(2) ___ Slide 2

(3) ___ Slide 3

(4) ___ Slide 4

(5) ___ It doesn't matter, her speed would be the same for each slide

By the work-energy theorem, the girl's speed at the bottom satisfies

$$\Delta K = W_{gravity}$$

or

$$\frac{1}{2}mv^2 - 0 = -mg\Delta y$$

which means v depends only on the height difference, not the shape of the slight

The correct answer is (5)

2. A bicycle wheel has a mass of 500 g and a radius of 33 cm. The wheel is set spinning at an initial angular velocity of 80 rev/min, and is then observed to slow to a complete halt over the next 45 s. Assume that the wheel undergoes constant angular acceleration during this time, and neglect the mass of the hub and spokes.

[max. 4 pts] (a) Through what number of revolutions does the wheel turn while it is slowing to a halt?

[max. 6 pts] (b) What constant torque must act on the wheel to account for its dynamics?

(a) Since the wheel undergoes constant angular acceleration, we can use

$$\begin{aligned}\Delta\phi &= \tfrac{1}{2}(\omega_0+\omega)t \\ &= \tfrac{1}{2}\left(80+0\ \tfrac{\text{rev}}{\text{min}}\right)(45\,\text{s}) \times \left(\tfrac{1\,\text{min}}{60\,\text{s}}\right) \\ &= 30\ \text{rev}\end{aligned}$$

(b) Applied torque has magnitude

$$|\tau| = I\,|\alpha| \qquad (1)$$

where for constant angular acceleration,

$$\alpha = \frac{\omega-\omega_0}{t}$$

and for a mass m all concentrated at radius R,

$$I = mR^2$$

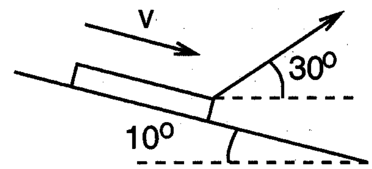
$$\begin{aligned}\Rightarrow\quad |\tau| &= mR^2\frac{|\omega-\omega_0|}{t} \\ &= (0.50\ \text{kg})(0.33\,\text{m})^2\ \frac{(80\ \text{rev/min})}{45\,\text{s}} \\ &\quad \times \left(\frac{2\pi}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) \\ &= 0.010\ \text{N m}\end{aligned}$$

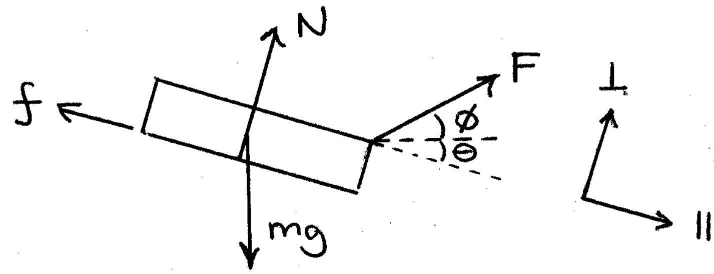
Note: α must be expressed in rad/time^2 for (1) to be valid

3. A man pulls a box of mass 30 kg using a taut rope oriented at 30° above the horizontal. The box slides down a slope, which makes an angle 10° with the horizontal. The box has a constant velocity $v = 1.5\,\mathrm{m/s}$. The coefficient of kinetic friction between the box and the ground is 0.30.

[max. 6 pts] (a) What is the magnitude of the force applied by the man to the box?

[max. 4 pts] (b) How much work does the man do on the box in a 20-second period?





(a) The box is in equilibrium

$$\sum F_{\parallel} = 0 \Rightarrow F\cos(\phi+\theta) + mg\sin\theta - f = 0 \qquad ①$$

$$\sum F_{\perp} = 0 \Rightarrow F\sin(\phi+\theta) + N - mg\cos\theta = 0 \qquad ②$$

Kinetic friction: $\qquad f = \mu_k N \qquad ③$

Combining ① - ③

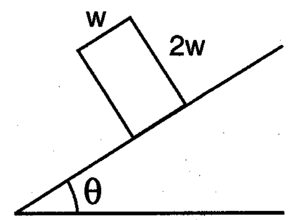
$$F\cos(\phi+\theta) + mg\sin\theta - \cancel{\mu_k N}$$
$$+ \mu_k F\sin(\phi+\theta) + \cancel{\mu_k N} - \mu_k mg\cos\theta = 0$$

$$F = \frac{\mu_k\cos\theta - \sin\theta}{\cos(\phi+\theta) + \mu_k\sin(\phi+\theta)}\, mg$$

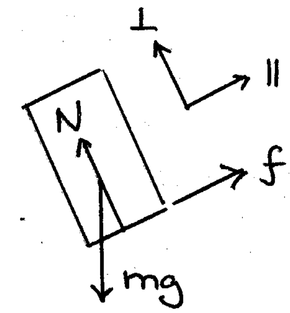
$$= 38\ \mathrm{N}$$

(b) $$W = \int \vec{F}\cdot d\vec{s}$$
$$= F v t \cos(\phi+\theta)$$
$$= 880\ \mathrm{J}$$

4. A cuboidal block twice as high as it is wide rests on a sloping board. The coefficient of static friction between block and board is 0.6. The slope of the board—and, hence, the magnitude of the angle θ in the diagram—is slowly increased. Will the block first tip over or begin to slide? (You must fully explain your reasoning to receive credit on this problem. No points will be awarded for just writing down a final answer that happens to be correct.)



So long as the box's bottom face is in contact with the board, the free-body diagram is as shown at right.



For equilibrium

$$\sum F_\perp = 0 \Rightarrow \qquad N = mg\cos\theta$$

$$\sum F_\parallel = 0 \Rightarrow \qquad f = mg\sin\theta$$

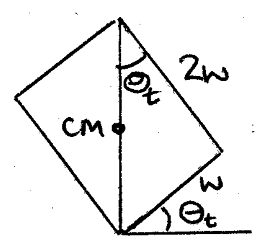
But $f \le \mu_s N$, so the critical angle at which the box begins to slip satisfies

$$\mu_s mg\cos\theta_s = mg\sin\theta_s$$

$$\tan\theta_s = \mu_s$$

$$\theta_s = \tan^{-1}\mu_s = 31.0^\circ$$

The block will tip about its bottom edge once the CM moves to the left of the pivot along that edge. The critical angle is determined by the diagram at right.



$$\tan\theta_t = \frac{w}{2w} = \frac{1}{2}$$

$$\theta_t = \tan^{-1}\tfrac{1}{2} = 26.5^\circ$$

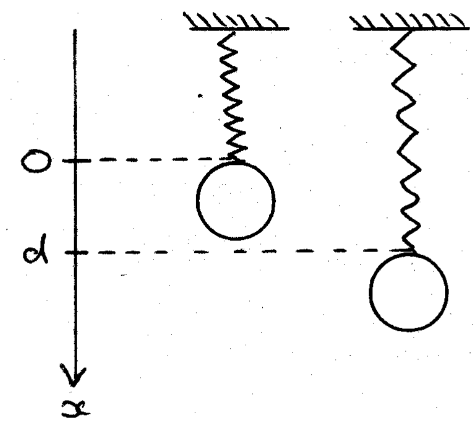
Since $\theta_t < \theta_s$, the block will tip first.

5. Real springs exhibit deviations from Hooke's law at large extensions. Suppose that a particular spring is massless and obeys the force-extension relation $F(x) = -kx + \alpha x^2$ for $x \geq 0$, where k and α are both positive. One end of the spring is attached to a rigid support, and the other is attached to a point mass m. The mass is initially supported in a position such that the spring is unstretched and hanging vertically. Then the mass is released.

[max. 5 pts] (a) Find the work done by the spring during the time that the mass descends a distance d.

[max. 5 pts] (b) Find the mass's velocity as a function of d.

(a)
$$\begin{aligned} W_s &= \int \vec{F} \cdot d\vec{s} \\ &= \int_0^d F(x)\,dx \\ &= \int_0^d (-kx + \alpha x^2)\,dx \\ &= \left[-\tfrac{1}{2}kx^2 + \tfrac{1}{3}\alpha x^3\right]_0^d \\ &= -\tfrac{1}{2}kd^2 + \tfrac{1}{3}\alpha d^3 \end{aligned}$$



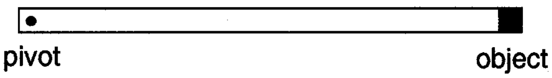
(b) By the work-energy theorem

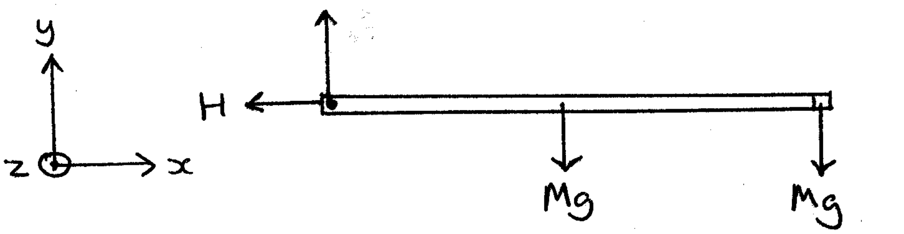
$$\Delta K = W = W_{grav} + W_s$$

$$\tfrac{1}{2}mv^2 = mgd - \tfrac{1}{2}kd^2 + \tfrac{1}{3}\alpha d^3$$

$$v = \sqrt{2gd - \frac{kd^2}{m} + \frac{2\alpha d^3}{3m}}$$ (directed downward)

6. A thin, uniform stick of length L and mass M is pivoted at one end about a fixed, horizontal axis, so that it can swing freely in a vertical plane. A small, metal object, also of mass M, is glued to the other end of the stick. The stick is held horizontally and released from rest. What is the initial acceleration of the metal object?





Taking torques about the pivot,

$$\sum \tau_z = I\alpha_z \Rightarrow -\left(Mg\frac{L}{2} + MgL\right) = \Big(\underbrace{\tfrac{1}{3}ML^2}_{\text{rod about end}} + \underbrace{ML^2}_{\text{point mass}}\Big)\alpha_z$$

$$-\frac{3}{2}MgL = \frac{4}{3}ML^2\alpha_z$$

$$\alpha_z = -\frac{9g}{8L}$$

The metal object is rotating in a circle of radius L about the pivot

$\Rightarrow$ Its acceleration has components

$$a_x = -m\omega^2 L = 0$$

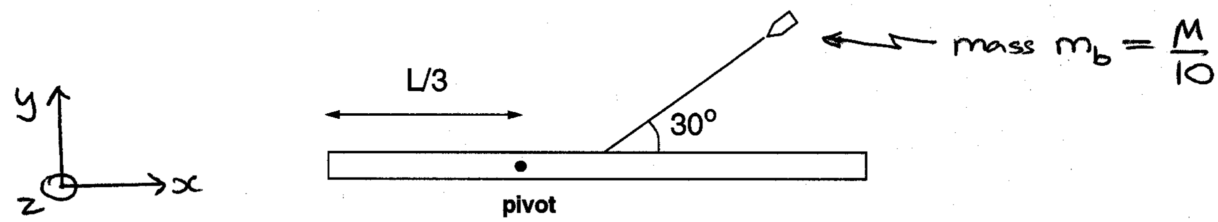
$$a_y = L\alpha_z = -\frac{9}{8}g$$

$$a_z = 0$$

So the magnitude of the initial acceleration is

$$a = \frac{9g}{8}$$

7. A thin, uniform rod of length L and mass M can rotate freely in a horizontal plane around a point $L/3$ from one end. A bullet of mass $M/10$, traveling at velocity v at an angle of 30° to the rod, hits the rod at its center of mass. The bullet embeds itself in the rod. What is the resultant angular velocity of the rod?



The combined system of rod plus bullet has a conserved angular momentum about the pivot.

Initially,
$$L_{i,z} = m_b v \underbrace{\left(\frac{L}{2} - \frac{L}{3}\right)}_{\text{distance from pivot to point of impact}} \sin 30°$$

$$= m_b v \frac{L}{6} \cdot \frac{1}{2} = \frac{m_b v L}{12}$$

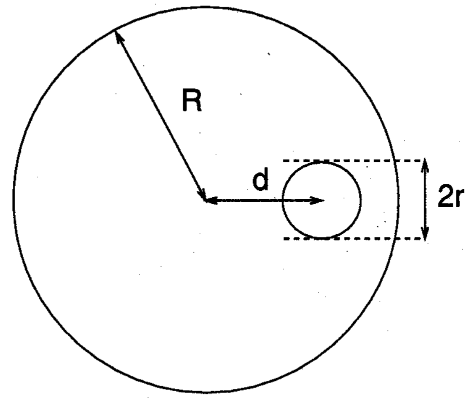
Finally,
$$L_{f,z} = (I_{rod} + I_{bullet})\omega$$
$$= \left[\underbrace{\frac{1}{12}ML^2 + M\left(\frac{L}{6}\right)^2}_{\text{parallel-axis theorem}} + m_b\left(\frac{L}{6}\right)^2\right]\omega$$
$$= (4M + m_b)\frac{L^2\omega}{36}$$

$L_{f,z} = L_{i,z} \Rightarrow \quad \omega = \dfrac{3m_b v}{(4M+m_b)L}$

$M = 10m_b \Rightarrow \quad \omega = \dfrac{3v}{41L}$

Strictly, the answer should be negative (for a counterclockwise rotation)

8. A rigid body has the external shape of a sphere of radius R. Its interior has a uniform density ρ, apart from a spherical cavity, which has a radius r and is centered a distance d from the center of the larger sphere. The interior of the cavity contains no matter (i.e., it is a perfect vacuum). Find the rotational inertia of this body about an axis that passes through its center of mass and is oriented perpendicular to the line connecting the center of the larger sphere to the center of the cavity, i.e., the axis is normal to the plane of the diagram.



The CM of the body will lie on the straight line that passes through the center of the two spheres. Take the origin to be at the center of the larger sphere.

Then
$$M_{\substack{\text{solid}\\\text{sphere}}} \cdot 0 = M_{\text{body}}\, x_{cm} + M_{\substack{\text{filled}\\\text{cavity}}}\, d$$

$$x_{cm} = -\frac{M_{\text{filled cavity}}}{M_{\text{body}}}\, d$$

$$= -\frac{\frac{4\pi}{3}\rho r^3}{\frac{4\pi}{3}\rho(R^3-r^3)}\, d = -\frac{r^3 d}{R^3-r^3}$$

Also
$$I_{\text{solid sphere}} = I_{\text{body}} + I_{\text{filled cavity}}$$

By the parallel-axis theorem,

$$I_{\text{solid sphere}} = \frac{2}{5} M_{\substack{\text{solid}\\\text{sphere}}} R^2 + M_{\substack{\text{solid}\\\text{sphere}}}\, x_{cm}^2$$

$$= \frac{4\pi R^3}{3}\rho\left(\frac{2}{5}R^2 + x_{cm}^2\right)$$

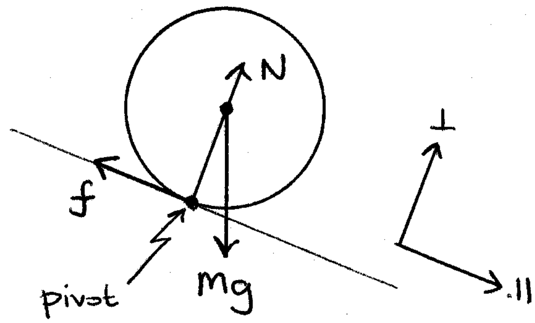
$$I_{\text{filled cavity}} = \frac{2}{5} M_{\substack{\text{filled}\\\text{cavity}}} r^2 + M_{\substack{\text{filled}\\\text{cavity}}} (d - x_{cm})^2$$

$$= \frac{4\pi r^3}{3}\rho\left[\frac{2}{5}r^2 + (d-x_{cm})^2\right]$$

$$I_{\text{body}} = \frac{8\pi\rho}{15}(R^5-r^5) + \frac{4\pi\rho d^2}{3}\,\frac{R^3 r^6 - R^6 r^3}{(R^3-r^3)^2}$$

$$= \frac{8\pi\rho}{15}(R^5-r^5) - \frac{4\pi\rho R^3 r^3 d^2}{3(R^3-r^3)}$$

9. A uniform solid cylinder has radius R, length L, and density ρ. When placed on a plane inclined at angle θ to the horizontal, the cylinder is observed to roll without slipping. Based on this information, what is the range of possible values for the coefficient of static friction between the cylinder and the plane?



$$\sum F_{\parallel} = ma \Rightarrow mg\sin\theta - f = ma \quad (1)$$

$$\sum F_{\perp} = 0 \Rightarrow N - mg\cos\theta = 0 \quad (2)$$

$$\sum \tau_z = I\alpha \Rightarrow -mgR\sin\theta = \left(\tfrac{1}{2}mR^2 + mR^2\right)\alpha$$

about instantaneous pivot point

parallel-axis theorem

$$= \tfrac{3}{2}mR^2\cdot\left(-\tfrac{a}{R}\right) \quad (3)$$

no-slip condition

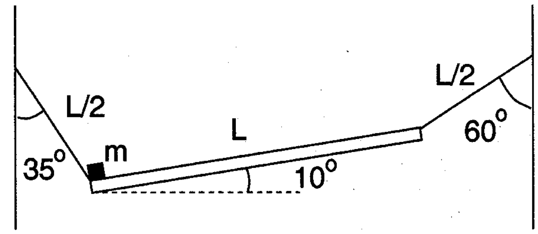
$$(3) \Rightarrow a = \tfrac{2}{3}g\sin\theta$$

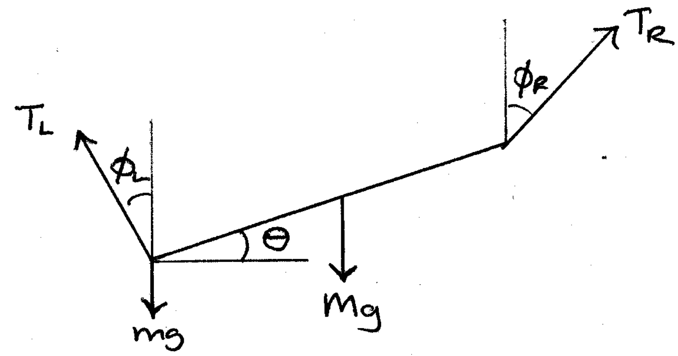
$$(1) \Rightarrow f = mg\sin\theta - ma$$
$$= \tfrac{1}{3}mg\sin\theta$$

But $f \le \mu_s N = \mu_s mg\cos\theta$ by (2)

$$\Rightarrow \tfrac{1}{3}mg\sin\theta \le \mu_s mg\cos\theta$$
$$\mu_s \ge \tfrac{1}{3}\tan\theta$$

10. A uniform board of mass M and length L is suspended between two vertical walls by two ideal ropes, each of length $L/2$. When a point mass m is placed at its left end, the board assumes the orientation shown in the diagram. Find the point mass m in terms of the board mass M.





In equilibrium

$$\Sigma F_x = 0 \Rightarrow T_R \sin\phi_R - T_L \sin\phi_L = 0 \quad (1)$$

$$\Sigma F_y = 0 \Rightarrow T_L \cos\phi_L + T_R \cos\phi_R - (M+m)g = 0 \quad (2)$$

$$\Sigma \tau_z = 0 \Rightarrow T_R L\cos(\phi_R+\theta) - Mg\frac{L}{2}\cos\theta = 0 \quad (3)$$

about point mass

$$(3) \Rightarrow T_R = \frac{1}{2}Mg\cos\theta\sec(\phi_R+\theta) = 1.44Mg$$

$$(1) \Rightarrow T_L = T_R \sin\phi_R \csc\phi_L = 2.17Mg$$

$$(2) \Rightarrow m = \frac{T_L\cos\phi_L + T_R\cos\phi_R}{g} - M = 1.50M$$

(Note: The ropes are attached to the walls at different heights.)