PHY 2060 Fall 2006 – Final Exam

1. The Savuka gold mine in South Africa extends 3.8 km below the Earth's surface. Find the fractional difference $\Delta g/g$ between the acceleration due to gravity at the mine's bottom and that at the surface. Assume for the purposes of this question that the Earth is a uniform sphere of radius 6,400 km.

According to the shell theorems, the acceleration due to gravity at a distance r from the Earth's center is

$$g(r) = \frac{GM(r)}{r^2}$$

where $$M(r) = \frac{4\pi\rho}{3} r^3$$

is the mass of that part of the Earth lying within distance r of the center

$$\Rightarrow \quad g(r) = \frac{4\pi\rho G}{3} r$$

$$\Delta g = g(R-d) - g(R)$$

$$= -\frac{4\pi\rho G}{3} d$$

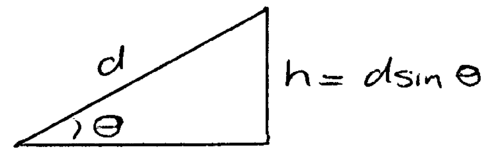
$$\frac{\Delta g}{g} = -\frac{d}{R}$$

$$= -\frac{3.8\,\text{km}}{6400\,\text{km}}$$

$$= -5.9 \times 10^{-4}$$

2. The engine of a 1500-kg car delivers 30 kW of useful power to its drive wheels. Find the maximum road speed at which the car can climb a 5° incline. Neglect air resistance.

Method I:



Useful power = rate of increase of gravitational potential energy

$$P = \frac{d}{dt}(mgh) = mg\frac{dh}{dt}$$

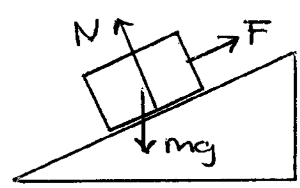
$$= mg\,v\sin\theta$$

$$v = \frac{P}{mg\sin\theta}$$

$$= \frac{30\times10^3\ \mathrm{W}}{(1500\ \mathrm{kg})(10\ \mathrm{m/s^2})(\sin 5°)}$$

$$= 23\ \mathrm{m/s}$$

Method I:



For equilibrium (constant velocity), the driving force F must obey

$$F = mg\sin\theta$$

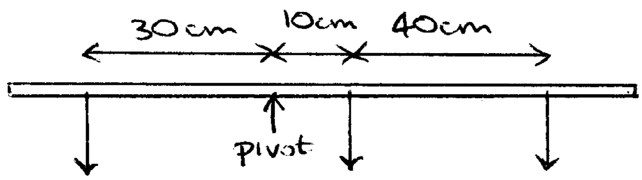
$$P = \vec{F}\cdot\vec{v}$$

$$= Fv \qquad \text{since } \vec{F} \parallel \vec{v}.$$

$$v = \frac{P}{F} = \frac{P}{mg\sin\theta}$$

$$= 23\ \mathrm{m/s} \qquad \text{as before}$$

3. A meter rule has a 0.6-kg mass hanging from the 90-cm mark and a 1.1-kg mass hanging from the 10-cm mark. You find that the rule balances in a horizontal orientation when supported at the 40-cm mark. What is the mass of the meter rule?



The weight of the rule acts at its CM, i.e., at the 50-cm mark.

For static equilibrium, require

$$0 = \sum_i \tau_i$$

$$= \left[(1.1\,\text{kg})(0.30\,\text{m}) - m(0.10\,\text{m}) - (0.6\,\text{kg})(0.50\,\text{m})\right] g$$

$$m = \frac{(1.1\,\text{kg})(0.30\,\text{m}) - (0.60\,\text{kg})(0.50\,\text{m})}{0.10\,\text{m}}$$

$$= 0.3\,\text{kg}$$

4. A printing company attempts to increase profitability by speeding up its printing presses. One press is printing "Go Gators" labels that will later be stuck onto badges. Paper rolls at speed $0.8c$ past a circular stamp of diameter 2 inches, which descends instantaneously, prints a label, and then rises instantaneously to be ready to print the next label. However, when the press is stopped, the labels have to be thrown out because in their own rest frame they are oval-shaped, not circular.

[max 3 pts] (a) What is the dimension of each label in the direction at right angles to its motion during printing?

[max 4 pts] (b) What is the dimension of each label in the direction parallel to its motion during printing?

Give both your answers in the rest frame of the labels.

(a) There is no relativistic length change in directions perpendicular to the relative motion between reference frames.

$\Rightarrow$ width of label $= 2$ in

(b) In the rest frame of the stamp, the moving papers is length-contracted from L_0 to

$$L = L_0/\gamma$$

In the rest frame of the paper, the front and back of the stamp come down at different times.

In either frame, the conclusion is that once the labels come to rest relative to the stamp, they are longer than they are wide.

0 in stamp frame

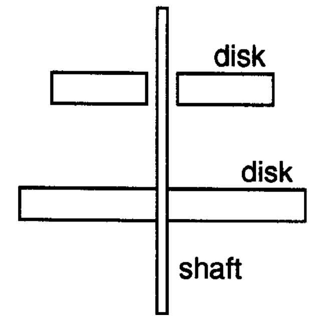
label frame $\rightarrow$ $\Delta x' = \gamma(\Delta x - \cancel{v\Delta t})$

$$= \frac{1}{\sqrt{1-(0.8)^2}}(2 \text{ in})$$

$$= \frac{5}{3}(2 \text{ in})$$

$$= 3\tfrac{1}{3} \text{ in}$$

5. The lower disk in the figure has mass 190 g and radius 2.5 cm. It is initially rotating at 150 rev/min on a light, frictionless shaft of negligible radius. The upper disk, of mass 110 g and radius 2.0 cm, is at rest. It is allowed to drop freely down the shaft onto the lower disk, and frictional forces act to bring the two disks to a common angular velocity. The disks are of uniform thickness and density.



   (a) What is the final angular velocity of the two disks?

   (b) What fraction of the initial kinetic energy is lost to friction?

(a) Angular momentum about the shaft is conserved:

$$L_f = L_i$$

$$(I_{upper} + I_{lower})\omega_f = I_{lower}\omega_i$$

where

$$I_j = \frac{1}{2} M_j R_j^2$$

$$I_{lower} = \frac{1}{2}(190\,g)(2.5\,cm)^2 = 594\,g\,cm^2$$

$$I_{upper} = \frac{1}{2}(110\,g)(2.0\,cm)^2 = 220\,g\,cm^2$$

$$\Rightarrow \quad \omega_f = \frac{594}{594+220}\, 150\ \text{rev/min}$$

$$= 110\ \text{rev/min} = 11.5\ \text{rad/s}$$

(b)

$$\frac{\Delta K}{K} = \frac{\frac{1}{2}(I_{upper}+I_{lower})\omega_f^2 - \frac{1}{2}I_{lower}\omega_i^2}{\frac{1}{2}I_{lower}\omega_i^2}$$

$$= \frac{(220+594)(110)^2 - (594)(150)^2}{(594)(150)^2}$$

$$= -0.27$$

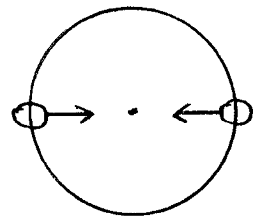
fractional loss of K.E.

$$\left|\frac{\Delta K}{K}\right| = 0.27$$

6. Two identical stars, each of mass M, rotate about their mutual center of mass in a common circular orbit of radius R. At all times, the stars are at diametrically opposite positions in the orbit.

   (a) Find the period of each star's orbit.

   (b) Find the total mechanical energy of this system (taking the gravitational potential energy to be zero at infinite separation).



(a) Gravitational force between stars distance $2R$ apart is

$$F_G = \frac{GM^2}{(2R)^2}$$

The centripetal force required to keep each star in uniform circular motion is

$$F_c = \frac{Mv^2}{R}$$

$$F_G = F_c \quad \Rightarrow \quad v^2 = \frac{GM}{4R}$$

$$T = \frac{2\pi R}{v}$$

$$= 4\pi\sqrt{\frac{R^3}{GM}}$$

(b) Total energy $\quad E = K + U$

Each star has kinetic energy

$$\tfrac{1}{2}Mv^2 = \frac{GM^2}{8R}$$

Gravitational potential energy of the combined system

$$U = -\frac{GM^2}{2R}$$

$$\Rightarrow \quad E = 2\frac{GM^2}{8R} - \frac{GM^2}{2R}$$

$$= -\frac{GM^2}{4R}$$

7. Jim throws a tennis ball at 55° to the horizontal so that it lands 32 m away on level ground. If Jim throws the tennis ball with the same initial speed and direction, how far away will it land on a 20° upward slope? Neglect air resistance.

The horizontal range is

$$R = \frac{v_0^2}{g} \sin 2\phi_0 = \frac{2v_0^2}{g} \sin\phi_0 \cos\phi_0$$

where v_0 = initial speed and ϕ_0 = launch angle.

The trajectory of a projectile (measured from a launch point at $x = y = 0$) is

$$y = x \tan\phi_0 - \frac{g}{2}\left(\frac{x}{v_0 \cos\phi_0}\right)^2$$

The equation for a slope making angle θ to the horizontal is

$$y = x \tan\theta$$

The landing point satisfies

$$x \tan\theta = x \tan\phi_0 - \frac{g}{2}\left(\frac{x}{v_0 \cos\phi_0}\right)^2$$

The solutions are $x = 0$ (the launch point) and

$$x = \frac{2v_0^2}{g} \cos^2\phi_0 \,(\tan\phi_0 - \tan\theta)$$

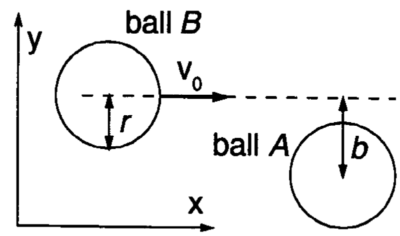
But $\frac{2v_0^2}{g} = R \csc\phi_0 \sec\phi_0$

$$\Rightarrow \quad x = R(1 - \tan\theta \cot\phi_0)$$

The distance from the launch point (i.e., the distance along the slope) is

$$\begin{aligned} d &= x \sec\theta \\ &= R \sec\theta \,(1 - \tan\theta \cot\phi_0) \\ &= 25.4 \text{ m} \end{aligned}$$

8. Ball A of radius r is at rest on a horizontal, frictionless surface when it is struck by an identical ball B traveling at speed v_0 parallel to the x axis (see diagram). The *impact parameter*—the shortest distance between the balls' centers if they were to pass through one another without colliding—is b. Find the final velocity (not just the speed) of each ball, assuming that the collision is elastic and the balls slide without rotating.

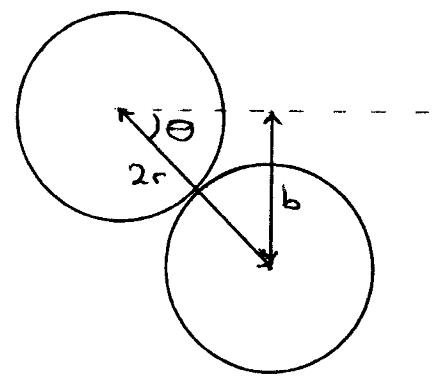


Hint: Consider the direction of the forces between the two balls. Also, you may find it helpful to recall that $\tan^2\theta + 1 = \sec^2\theta = 1/\cos^2\theta$.

Since A is initially stationary, its final velocity points along the direction of the contact force, i.e., at an angle

$$\theta = \sin^{-1}\frac{b}{2R}$$

below the $+x$ axis:



$$V_{Ax} = V_A\cos\theta$$
$$V_{Ay} = V_A\sin\theta$$

Linear momentum is conserved in the collision.

Along x: $mV_0 = mV_A\cos\theta + mV_{Bx} \Rightarrow V_{Bx} = V_0 - V_A\cos\theta$

Along y: $0 = -mV_A\sin\theta + mV_{By} \Rightarrow V_{By} = V_A\sin\theta$

In an elastic collision, kinetic energy is also conserved:

$$\tfrac{1}{2}mV_0^2 = \tfrac{1}{2}m_A V_A^2 + \tfrac{1}{2}mV_B^2$$

$$V_0^2 = V_A^2 + (V_0 - V_A\cos\theta)^2 + (V_0\sin\theta)^2$$

$$= V_A^2 + V_0^2 - 2V_0V_A\cos\theta + V_A^2(\cos^2\theta + \sin^2\theta)$$

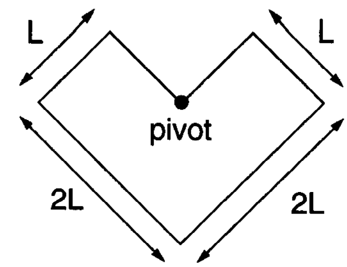
$$\cancel{V_0^2} = \cancel{V_0^2} + 2V_A(V_A - V_0\cos\theta)$$

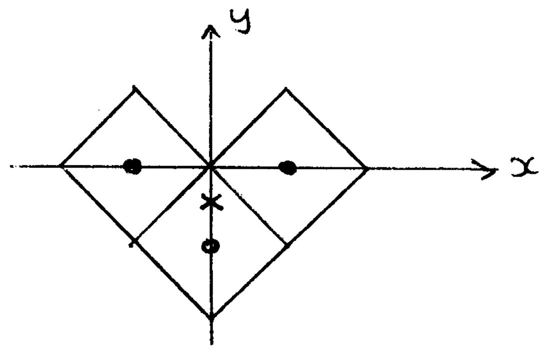
Thus $V_A = V_0\cos\theta$ where $\theta = \sin^{-1}\frac{b}{2r}$

$V_{Ax} = V_0\cos^2\theta$ $\quad V_{Ay} = -V_0\tan\theta$

$V_{Bx} = V_0\sin^2\theta$ $\quad V_{By} = V_0\tan\theta$

9. A thin, uniform plate is cut and suspended from a frictionless pivot as shown in the figure. Find the frequency of small oscillations about the plate's equilibrium position.





For a physical pendulum

$$f = \frac{1}{T} = 2\pi\sqrt{\frac{mgd}{I}}$$

Total mass

$$\begin{aligned} m &= (\text{mass}/\text{area}) \times (\text{area}) \\ &= \sigma \quad 3L^2 \\ &= 3\sigma L^2 \end{aligned}$$

Distance from pivot (origin in diagram above) to cm (x in diagram)

$$\begin{aligned} d &= -y_{cm} \\ &= -\frac{\frac{m}{3}\left(0 + 0 - L/\sqrt{2}\right)}{m} \\ &= -\frac{L}{3\sqrt{2}} \end{aligned}$$

Rotational inertia of plate about pivot

$$I = 3\Big[\underbrace{\tfrac{1}{6}m_\square L^2}_{\text{square plate about its cm}} + \underbrace{m_\square\left(\tfrac{L}{\sqrt{2}}\right)^2}_{\text{parallel axes theorem}}\Big]$$

$$= 3\, m_\square\left(\frac{L^2}{6} + \frac{L^2}{2}\right) \qquad m_\square = \frac{m}{3}$$

$$= \frac{2}{3} m L^2$$

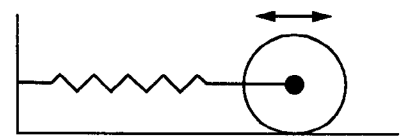
Thus

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{mg\, L/3\sqrt{2}}{2mL^2/3}} \\ &= \frac{1}{2\pi}\sqrt{\frac{g}{2\sqrt{2}L}} \end{aligned}$$

10. A solid cylinder of mass M and radius R is mounted on a frictionless axle passing through its center so that the cylinder can roll on a level tabletop. The axle, of mass m and negligible radius, is attached to one end of a horizontal, ideal spring of spring constant k. The other end of the spring is attached to a rigid support. Suppose that the axle is displaced horizontally a small distance from its equilibrium position, and is then released. In its subsequent motion, the cylinder rolls without slipping over the table.

(a) Write down the kinetic energy of this device as a function of the axle's velocity v.

(b) Use energy conservation to deduce the period of oscillation of the axle.



(a) $$K = K_{\text{translation}} + K_{\text{rotation}}$$
$$= \tfrac{1}{2}(M+m)v^2 + \tfrac{1}{2} I \omega^2$$

where $$I = \tfrac{1}{2} M R^2$$

and $$\omega = v/R \quad \text{(no-slip condition)}$$

$$\Rightarrow \quad K = \tfrac{1}{2}(M+m)v^2 + \tfrac{1}{4} M v^2$$
$$= \left(\tfrac{3}{4} M + \tfrac{1}{2} m\right) v^2$$

(b) Require $$E = K + U = \text{constant} \quad \text{(friction does no work)}$$
$$E = \left(\tfrac{3}{4} M + \tfrac{1}{2} m\right) v^2 + \tfrac{1}{2} k x^2 = \text{constant}$$

Look for a SHM solution
$$x = A\cos(\omega t + \phi)$$
$$v = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow \quad E = \left(\tfrac{3}{4} M + \tfrac{1}{2} m\right) \omega^2 A^2 \;(\omega t + \phi) + \tfrac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

This is constant (time-independent) if and only if

$$\left(\tfrac{3}{4} M + \tfrac{1}{2} m\right) \omega^2 = \tfrac{1}{2} k$$
$$\omega = \sqrt{\frac{2k}{3M+2m}}$$
$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{3M+2m}{2k}}$$