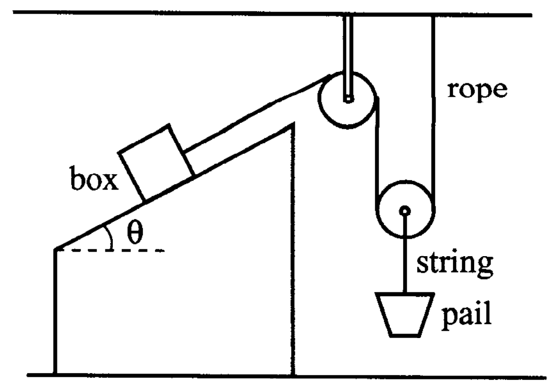
PHY 2060 Fall 2007 - Exam 2 Solution

1. The figure shows a box of mass m_b on a frictionless ramp inclined at an angle θ to the horizontal. An ideal rope runs from the box, around a fixed pulley and a suspended pulley, to a fixed attachment point on the ceiling. A pail of mass m_p hangs by an ideal string from the suspended pulley. The pulleys are massless and frictionless.



Place a check to the left of any/all of the following statements that **must** be true:

i. The tension in the string is $m_p g$.

✓ ii. The tension in the string is twice that in the rope.

✓ iii. The acceleration of the pail is half as great (in magnitude) as the acceleration of the box.

iv. The tension in the rope is $m_b g \cos\theta$.

v. The acceleration of the box must be smaller in magnitude than g (the free-fall acceleration due to gravity).

i) N2 applied to the pail gives

$$T_s - m_p g = m_p a_p$$
$$\neq 0 \quad \text{in general} \Rightarrow \text{FALSE.}$$

ii) N2 applied to the suspended pulley gives

$$2T_r - T_s = \overset{0}{\cancel{m_{pulley}}}\, a_{pulley}$$
$$= 0 \qquad \Rightarrow \text{TRUE.}$$

iii) When the pail falls a distance d, the vertical length of rope on each side of the suspended pulley increases by d, so the box must move $2d$ up the slope $\Rightarrow$ TRUE.

iv) N2 applied to the box (parallel to the slope) gives

$$T_r - m_b g \sin\theta = m_b a_b$$
$$\neq 0 \text{ in general}$$

There is no reason for $T_r = m_b g \cos\theta$ $\Rightarrow$ FALSE

v) In the limit $m_p \gg m_b$, expect $a_p \simeq -g$ (near free-fall) and since $|a_b| = 2|a_p|$, $a_b \simeq 2g$ $\Rightarrow$ FALSE

2. A stream of darts, each of mass 75 g and traveling horizontally at 60 m/s, strike a door at a rate of 6 darts per second. Half the darts bury themselves in a dartboard mounted on the door, while the other half strike metal and recoil elastically, traveling back in the direction they came from. Find the average horizontal force experienced by the door due to the impact of the darts.

Consider a system consisting of the stream of darts. By N2, the net force on the system satisfies

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

or, averaged over a time interval Δt,

$$\vec{F}_{avg} = \frac{\Delta \vec{P}}{\Delta t}$$

Each dart that buries itself in the dartboard has

$$\Delta \vec{p} = -m\vec{v}_i$$

while each dart that recoils elastically has

$$\Delta \vec{p} = m(-\vec{v}_i) - m\vec{v}_i = -2m\vec{v}_i.$$

Since there are three of each type of dart striking per second,

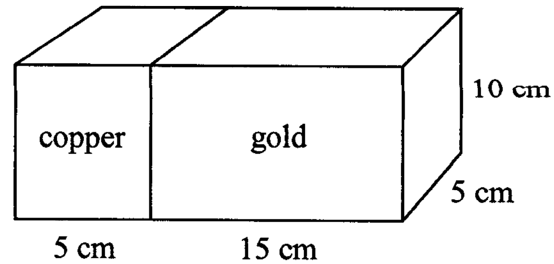
$$\vec{F}_{avg} = \frac{\Delta \vec{P}}{\Delta t} = 3(-m\vec{v}_i) + 3(-2mv_i)$$

$$= -40.5 \text{ kg m/s}$$

The force exerted by the darts on the door is

$$-\vec{F}_{avg} = 40.5 \text{ kg m/s}$$

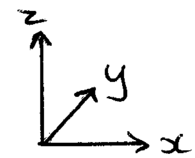
3. The figure shows a block composed of copper and gold slabs joined together. The density of copper is $8.96\,\mathrm{g/cm^3}$, and that of gold is $19.32\,\mathrm{g/cm^3}$. Find the Cartesian coordinates of the block's center of mass. Make sure you specify the origin of your coordinate system.



For the composite block formed from the two slabs,

$$\vec{r}_{cm} = \frac{m_{Cu}\vec{r}_{cm,Cu} + m_{Au}\vec{r}_{cm,Au}}{m_{Cu} + m_{Au}},$$

where symmetry dictates that the CM of each slab lies at the center of the cuboid.



Choose a set of Cartesian coordinates with its origin at the bottom left of the figure.

Then

$$\vec{r}_{cm,Cu} = (2.5, 2.5, 5.0)\ \mathrm{cm}$$

$$\vec{r}_{cm,Au} = (12.5, 2.5, 5.0)\ \mathrm{cm}$$

Also

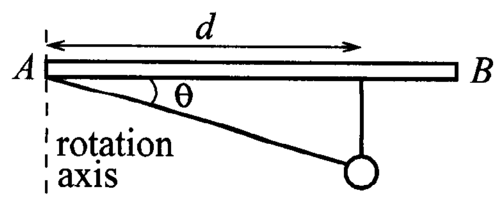
$$\begin{aligned} m_{Cu} &= \rho_{Cu} V_{Cu} \\ &= (8.96\,\mathrm{g/cm^3})(5 \times 5 \times 10\,\mathrm{cm^3}) \\ &= 2240\,\mathrm{g} \\ m_{Au} &= \rho_{Au} V_{Au} \\ &= (19.32\,\mathrm{g/cm^3})(15 \times 5 \times 10\,\mathrm{cm^3}) \\ &= 14490\,\mathrm{g} \end{aligned}$$

Substituting into the CM equation,

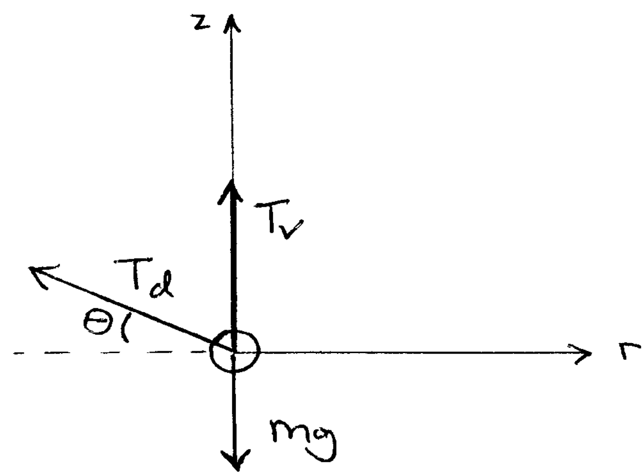
$$\vec{r}_{cm} = (11.16, 2.5, 5.0)\ \mathrm{cm}$$

The values of y_{cm} and z_{cm} are obvious by symmetry, but x_{cm} can only be found by direct calculation.

4. A point mass m is suspended from a horizontal rod by two ideal strings (as shown in the figure): a diagonal string that makes an angle θ to the rod, and is attached to the rod at the end marked A in the figure; and a vertical string that is attached to the rod a distance d from end A. Find the tension in the vertical string when the rod is rotated about the end A in such a way that the mass moves in a horizontal circle at a constant speed v.



Define a coordinate system rotating with the rod:



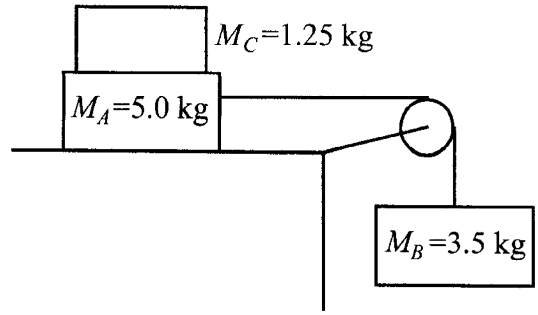
N2 along z: $T_d \sin\theta + T_v - mg = ma_z$
$$= 0$$

N2 along r: $-T_d \cos\theta = ma_r$
$$= -\frac{mv^2}{d} \text{ (centripetal)}$$

Combining these equations,

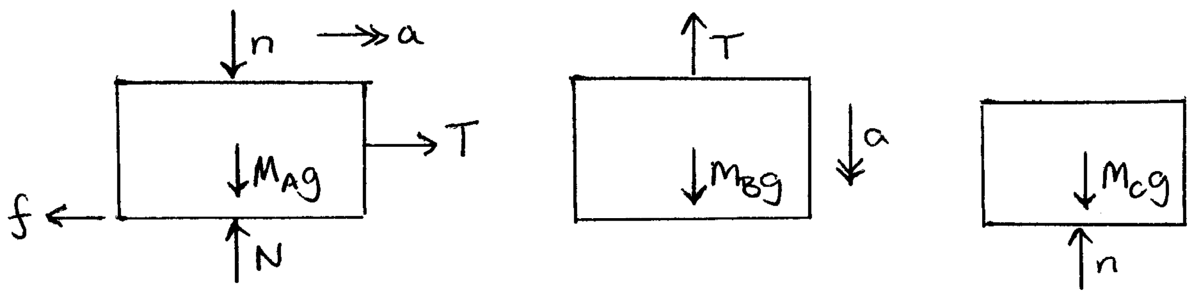
$$T_v = m\left(g - \frac{v^2}{d}\tan\theta\right)$$

5. Block A (mass 5.0 kg) rests on a horizontal table top. The coefficient of kinetic friction between A and the table is 0.35. Block B (mass 3.5 kg) is connected to A via an ideal string running over a massless, frictionless pulley, as shown in the figure. Block C (mass 1.25 kg), when placed on A, is just heavy enough to prevent the system from moving.



(a) Determine the coefficient of static friction between block A and the table.

(b) If block C is removed, what is the subsequent acceleration of block A?



Apply N2 to each mass:

C vertically:	$n - M_c g = 0$	$\Rightarrow n = M_c g$	①
B vertically:	$T - M_B g = -M_B a$	$\Rightarrow T = M_B(g-a)$	②
A vertically:	$N - n - M_A g = 0$	$\Rightarrow N = M_A g + n$	③
horizontally:	$T - f = M_A a$	$\Rightarrow T = M_A a + f$	④

(a) Here, $a = 0$ and $\quad f = \mu_s N = \mu_s (M_A + M_c) g$

Then ① - ④ $\Rightarrow$

$$\mu_s = \frac{M_B}{M_A + M_c} = 0.56.$$

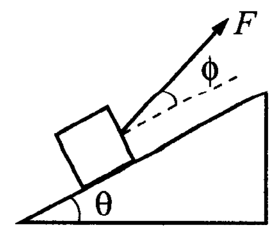
(b) We can still apply ① - ④ but now $M_c = 0$ and

$$f = \mu_k N = \mu_k M_A g$$

Then ① - ④ $\Rightarrow$

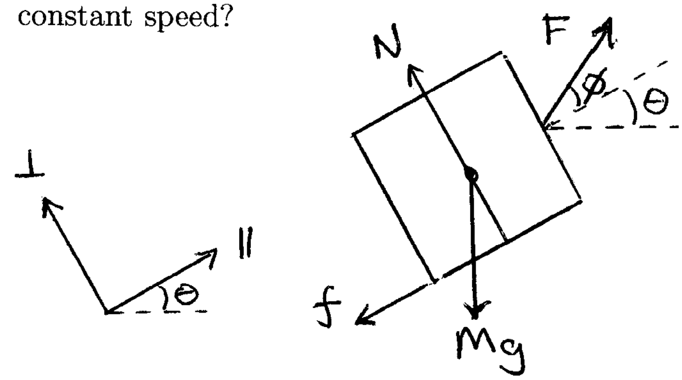
$$a = \frac{M_B - \mu_k M_A}{M_A + M_B} g = 2.1 \text{ m/s}^2$$

6. You wish to pull a box of mass M up an inclined plane at a constant speed by pulling with a force F on an ideal rope attached to the box. The plane makes an angle θ to the horizontal, and the coefficient of kinetic friction between the box and the plane is μ_k. You are free to choose the angle ϕ between the rope and the plane, and hence the angle $\theta + \phi$ between the rope and the horizontal (see the figure).



(a) Find an expression for F as a function of ϕ.

(b) What value of ϕ minimizes the force F that you must apply to raise the box at constant speed?



(a) Applying N2 ...

along ⊥: $N + F\sin\phi - Mg\cos\theta = 0$

$$N = Mg\cos\theta - F\sin\phi \quad (1)$$

along ||: $F\cos\phi - Mg\sin\theta - f = Ma = 0 \quad (2)$

Kinetic friction $\quad f = \mu_k N \quad (3)$

Combining (1)-(3),

$$F = Mg\,\frac{\sin\theta + \mu_k\cos\theta}{\cos\phi + \mu_k\sin\phi} \quad (4)$$

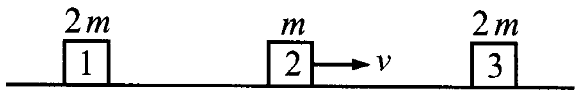
(b) Since ϕ enters only the denominator of (4), we can minimize F by maximizing

$$D = \cos\phi + \mu_k\sin\phi$$

$$0 = \frac{dD}{d\phi} = -\sin\phi + \mu_k\cos\phi$$

$$\phi = \tan^{-1}\mu_k$$

7. Three carts are spaced out as shown in the figure along a straight track that permits the carts to move (without friction) in only one dimension. Carts 1, 2, and 3 have masses $2m$, m, and $2m$, respectively. Carts 1 and 3 are initially stationary, while cart 2 is initially moving towards cart 3 at speed v, as shown in the figure. Find the final velocity of each cart, assuming that all collisions are elastic.



- First, consider the elastic collision between carts 2 and 3:

$$V_{2i} = V \qquad\qquad V_{3i} = 0$$

Using the formulae for the final velocities in 1D collisions,

$$V_{2f} = \frac{m_2 - m_3}{m_2 + m_3} V_{2i} + \frac{2m_3}{m_2 + m_3} V_{3i} = -\frac{V}{3}$$

$$V_{3f} = \frac{m_3 - m_2}{m_3 + m_2} V_{3i} + \frac{2m_2}{m_3 + m_2} V_{2i} = \frac{2V}{3}$$

- Second, there is an elastic collision between carts 1 and 2:

$$V_{1i} = 0 \qquad\qquad V_{2i} = -\frac{V}{3}$$

$$\Rightarrow \quad V_{1f} = \frac{m_1 - m_2}{m_1 + m_2} V_{1i} + \frac{2m_2}{m_1 + m_2} V_{2i} = -\frac{2V}{9}$$

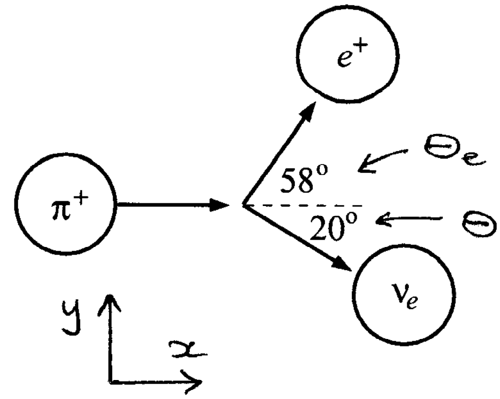
$$V_{2f} = \frac{m_2 - m_1}{m_2 + m_1} V_{2i} + \frac{2m_1}{m_2 + m_1} V_{1i} = \frac{V}{9}$$

- Since cart 2 is now moving to the right more slowly than cart 2, there isn't a third collision.

Thus, the final velocities are

$$V_1 = -\frac{2V}{9}, \quad V_2 = \frac{V}{9}, \quad V_3 = \frac{2V}{3}$$

8. A pion (π^+), with momentum 2.5×10^{-31} kg m/s, decays by emitting a positron (e^+) and an electron neutrino (ν_e) at angles of 58° and 20°, respectively, to the pion's initial direction of motion (see figure). What is each decay particle's velocity? For the purposes of this problem, assume that the mass of a positron is 0.511 MeV (9.15×10^{-31} kg) and the mass of an electron neutrino is 10.0 eV (1.79×10^{-35} kg). Ignore relativistic effects.



Conservation of linear momentum ...

along x: $m_\pi V_\pi = m_e V_e \cos\theta_e + m_\nu V_\nu \cos\theta_\nu$ (1)

along y: $0 = m_e V_e \sin\theta_e - m_\nu V_\nu \sin\theta_\nu$ (2)

(2) $\Rightarrow$ $m_\nu V_\nu = m_e V_e \sin\theta_e \csc\theta_\nu$

(1) $\Rightarrow$ $m_\pi V_\pi = m_e V_e (\cos\theta_e + \sin\theta_e \cot\theta_\nu)$

$$V_e = \frac{m_\pi V_\pi}{m_e(\cos\theta_e + \sin\theta_e \cot\theta_\nu)}$$

$$= \frac{P_\pi}{m_e(\cos\theta_e + \sin\theta_e \cot\theta_\nu)}$$

$$= 0.096 \text{ m/s}$$

$$V_\nu = \frac{m_e}{m_\nu} V_e \sin\theta_e \csc\theta_\nu$$

$$= 1.2 \times 10^4 \text{ m/s}$$

9. Two ice pucks A and B, of identical mass, are fired towards one another, each sliding at speed v_1 across horizontal, frictionless ice. They collide, and afterwards the two pucks travel off in opposite directions, each moving at speed $2v_1/3$. In a later experiment, puck B is stationary on the ice when puck A is launched towards it at speed v_2 along the positive x axis. After a collision, the motion of both pucks is confined to the x axis. Find the final velocities (magnitudes and directions along the x axis) of the two pucks.

Collisions between the pucks satisfy

$$|\vec{V}_{Bf} - \vec{V}_{Af}| = \varepsilon\, |\vec{V}_{Bi} - \vec{V}_{Ai}|$$

where ε is the coefficient of restitution.

For the first collision,

$$|\vec{V}_{Bi} - \vec{V}_{Ai}| = 2V_1$$

$$|\vec{V}_{Bf} - \vec{V}_{Af}| = 4V_1/3$$

$$\Rightarrow \qquad \varepsilon = \frac{4V_1/3}{2V_1} = \frac{2}{3}$$

For the second collision, we can use the 1D formula

$$V_{Af} = \frac{1}{m_A + m_B}\left[(m_A - \varepsilon m_B)V_{Ai} + m_B(1+\varepsilon)V_{Bi}\right]$$

$$= \frac{1}{2m}\left[m(1 - 2/3)V_2 + 0\right]$$

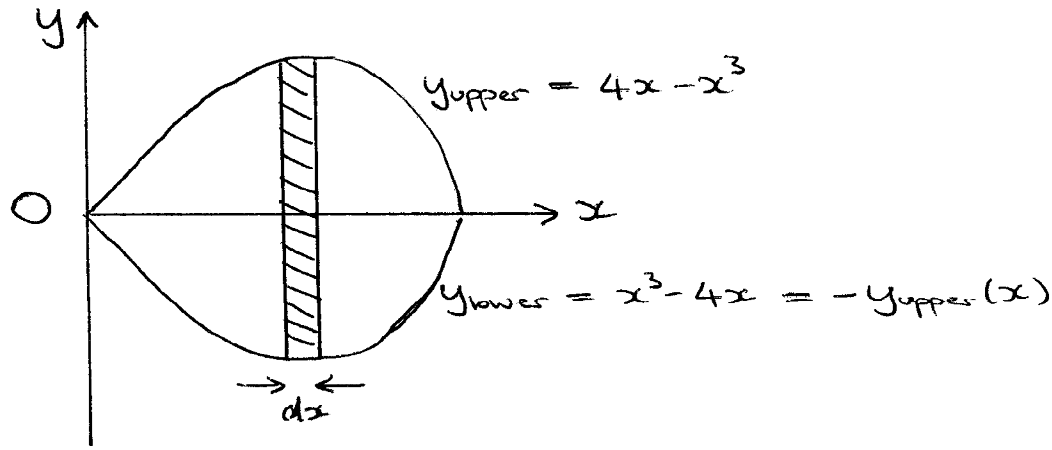
$$= \frac{V_2}{6}$$

$$V_{Bf} = \frac{1}{m_A + m_B}\left[m_A(1+\varepsilon)V_{Ai} + (m_B - \varepsilon m_A)V_{Bi}\right]$$

$$= \frac{1}{2m}\left[m\left(1+\tfrac{2}{3}\right)V_2 + 0\right]$$

$$= \frac{5V_2}{6}$$

10. A two-dimensional object is formed by the region of the x-y plane that (i) is enclosed by the curves $y = 4x - x^3$ and $y = x^3 - 4x$, and (ii) satisfies $x \geq 0$. Find the x coordinate of this object's center of mass.



The two curves intersect where

$$y_{upper}(x) = y_{lower}(x)$$
$$4x - x^3 = x^3 - 4x$$
$$x(x^2 - 4) = 0$$
$$x = 0, \pm 2$$

Therefore, the shape is confined to the region $0 \leq x \leq 2$.

Area:

$$A = \int_0^2 dx \left[y_{upper}(x) - y_{lower}(x)\right]$$
$$= 2\int_0^2 dx\,(4x - x^3)$$
$$= 2\left[2x^2 - \frac{1}{4}x^4\right]_0^2$$
$$= 8$$

Center of mass:

$$x_{cm} = \frac{1}{A}\int_0^2 dx\; x\left[y_{upper}(x) - y_{lower}(x)\right]$$
$$= \frac{2}{A}\int_0^2 dx\,(4x^2 - x^4)$$
$$= \frac{1}{4}\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= \frac{16}{15}$$