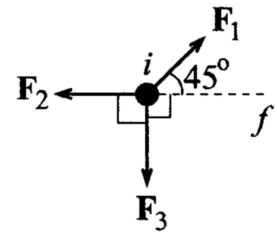
PHY 2060 Fall 2007 – Exam 3 Solution

1. A particle of mass m is subjected to just three constant forces $\mathbf{F}_j$ (j = 1, 2, 3), oriented along the directions shown in the figure. Starting at rest, the particle moves off along the straight path indicated by the dashed line. Let W_j be the work done by force j on the particle during the displacement of the particle from the position labeled i to the position labeled f.



Place a check to the left of any/all of the following statements that **must** be true:

(a) $-W_1 > W_2 > W_3$.

✓ (b) $W_1 > W_3 > W_2$.

(c) $-W_2 > W_1 > W_3$.

(d) $W_1 > W_2 > W_3$.

✓ (e) $W_1 > -W_2 > W_3$.

Work done by force $\vec{F}_j$ is

$$W_j = \int_i^f \vec{F}_j \cdot d\vec{s}.$$

$$W_1 = F_1 s \cos 45° = F_1 s/\sqrt{2}$$

$$W_2 = F_2 s \cos 180° = -F_2 s$$

$$W_3 = F_3 s \cos 90° = 0.$$

Applying N2 to the particle, resolved along x:

$$F_1/\sqrt{2} - F_2 = ma_x > 0$$

$$\Rightarrow \qquad F_1/\sqrt{2} > F_2$$

$$\Rightarrow \qquad W_1 > -W_2$$

Thus, (b) and (e) are TRUE; (a), (c) and (d) are FALSE

2. A bicycle has wheels whose diameter (including the tires) is 66 cm. The bicycle chain runs around a 39-tooth gear wheel attached to the crankshaft and a 24-tooth gear attached to the hub of the rear wheel. The pedals are also mounted on the crankshaft at the end of 17-cm-long crank arms. Calculate the bike's speed (assuming the tires do not slip on the ground) when the pedals are moving at 0.50 m/s relative to the bike's frame.

The connection between linear and angular velocities is

$$V = R\omega.$$

$$\Rightarrow \quad V_{bike} = R_{wheel}\,\omega_{wheel},$$

$$V_{pedal} = R_{crank\ arm}\,\omega_{crank}$$

Also, since the circumference of each gear must move at the same speed (the speed of the chain)

$$\frac{\omega_{wheel}}{\omega_{crank}} = \frac{(\#\text{ teeth})_{crank}}{(\#\text{ teeth})_{wheel}}$$

Thus,

$$V_{bike} = R_{wheel}\,\frac{(\#\text{ teeth})_{crank}}{(\#\text{ teeth})_{wheel}}\,\frac{V_{pedal}}{R_{crank\ arm}}$$

$$= \frac{0.66\text{m}}{2}\;\frac{39}{24}\;\frac{0.50\text{m/s}}{0.17\text{ m}}$$

$$= 1.6\text{ m/s}$$

3. A nucleus at rest decays into three particles. Two of the particles are directly detected: particle 1, having a mass of 15.4×10^{-27} kg, travels due west at 7.82×10^{6} m/s; and particle 2, having a mass of 9.20×10^{-27} kg, travels due south at 4.39×10^{6} m/s. All that is known about particle 3 is that its mass is 11.9×10^{-27} kg.

(a) What is the momentum (magnitude and direction) of particle 3?

(b) How much kinetic energy is created during this decay?

(a) By conservation of linear momentum

$$\vec{P}_f = \vec{P}_i$$

$$\sum_j \vec{P}_j = \vec{0}$$

$$P_{3x} = -m_1 V_{1x} - P_2 V_{2x}$$

$$= -(15.4 \times 10^{-27}\,\text{kg})(-7.82 \times 10^{6}\,\text{m/s}) - (9.20 \times 10^{-27}\,\text{kg})(0)$$

$$= 1.20 \times 10^{-19}\,\text{kg m/s.}$$

$$P_{3y} = -m_1 V_{1y} - m_2 V_{2y}$$

$$= -(15.4 \times 10^{-27}\,\text{kg})(0) - (9.20 \times 10^{-27}\,\text{kg})(-4.39 \times 10^{6}\,\text{m/s})$$

$$= 4.04 \times 10^{-20}\,\text{kg m/s.}$$

magnitude: $$|\vec{P}_3| = \sqrt{P_{3x}^2 + P_{3y}^2} = 1.27 \times 10^{-19}\,\text{kg m/s}$$

angle CCW from due east:

$$\theta = \tan^{-1}\frac{P_{3y}}{P_{3x}} = 18.6^\circ$$

(b) For each particle, the kinetic energy is

$$K_j = \tfrac{1}{2} m_j v_j^2 \equiv \frac{P_j^2}{2m_j}$$

Total K.E. $$K = \tfrac{1}{2}(15.4 \times 10^{-27}\,\text{kg})(7.82 \times 10^{6}\,\text{m/s})^2 + \tfrac{1}{2}(9.20 \times 10^{-27}\,\text{kg})(4.39 \times 10^{6}\,\text{m/s})^2 + \frac{(1.27 \times 10^{-19}\,\text{kg m/s})^2}{2(11.9 \times 10^{-27}\,\text{kg})}$$

$$= 1.24 \times 10^{-12}\,\text{J}$$

4. The assumption that the gravitational force of a mass m at any point near the Earth's surface is of magnitude mg is only an approximation. A better approximation for the gravitational force at a height h above sea level is $\mathbf{F}_g = -mg(1 - 2h/R_E)\hat{\jmath}$, where $\hat{\jmath}$ is a unit vector pointing vertically upward and $R_E = 6,400\,\text{km}$ is the mean radius of the Earth.

   (a) Obtain an expression for the work done by the gravitational force when the point of application of the force moves from height h_i to height h_f.

   (b) Calculate the work done by gravity on a bottle of Coors beer when it is transported from the brewery in Golden, CO (1,730 m above sea level) to the Orange and Brew in the Reitz Union (40 m above sea level). Take the mass of the bottle and its contents to be 530 g.

   (c) By what percentage would your answer to (b) be different if you were to use the standard approximation $\mathbf{F}_g = -mg\hat{\jmath}$?

(a) Work done by gravity

$$
\begin{aligned}
W_g &= \int_i^f \vec{F}_g \cdot d\vec{s} \\
&= -mg \int_{h_i}^{h_f} \left(1 - \frac{2h}{R_E}\right) dh = -mg\left[h - \frac{h^2}{R_E}\right]_{h_i}^{h_f} \\
&= -mg\left(h_f - h_i - \frac{h_f^2 - h_i^2}{R_E}\right) \\
&= -mg(h_f - h_i)\left(1 - \frac{h_f + h_i}{R_E}\right)
\end{aligned}
$$

(b)

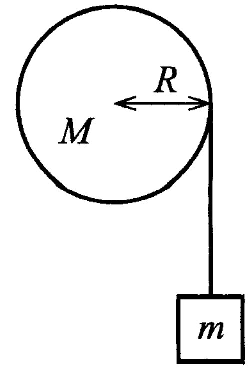
$$
\begin{aligned}
W_g &= -(0.530\,\text{kg})(10\,\text{m/s}^2)(40\,\text{m} - 1730\,\text{m}) \\
&\quad \times \left(1 - \frac{40\,\text{m} + 1730\,\text{m}}{6.4 \times 10^6\,\text{m}}\right) \\
&= 8950\,\text{J}
\end{aligned}
$$

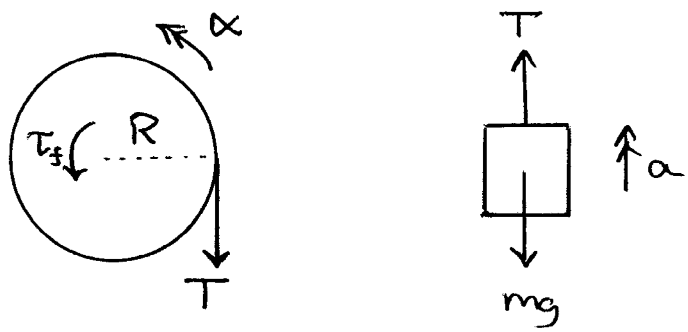
(c)

$$
\left|\frac{\Delta W_g}{W_g}\right| = \frac{h_f + h_i}{R_E} = \frac{40\,\text{m} + 1730\,\text{m}}{6.4 \times 10^6\,\text{m}}
$$

$$
= 2.8 \times 10^{-4} \equiv 0.028\%
$$

5. A uniform flywheel of radius R and mass M pivots about a fixed, massless axle. An ideal string is wrapped around the rim of the flywheel. A box of mass m hangs off the free end of the string, as shown in the figure. When the axle rotates, it rubs against its bearings, creating a frictional torque of magnitude τ_f (measured with respect to the common axis of rotational symmetry of the axle and the flywheel). Find the box's acceleration once this system is in motion.





For the flywheel, taking torques about the axis of rotation,

$$\tau_f - RT = I\alpha \qquad (1)$$

where $I = \frac{1}{2}MR^2$ and $\alpha = \frac{a}{R} < 0$.

For the box,

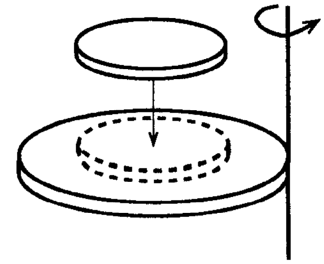
$$T - mg = ma. \qquad (2)$$

(1) + R(2) $\Rightarrow$

$$\tau_f - mgR = \left(\tfrac{1}{2}MR + mR\right)a$$

$$a = -\frac{mg - \tau_f/R}{m + \frac{1}{2}M} \quad \left(\text{directed downward}\right)$$

6. A disk of mass $M = 1.29\,\mathrm{kg}$ and radius $R = 3.59\,\mathrm{m}$ is attached at a point on its circumference to an ideal (massless, zero-radius, frictionless) axle. The axle is oriented vertically, and the plane of the disk is horizontal, as shown in the figure. This disk is rotating at an angular speed of $539\,\mathrm{rev/minute}$ when a second disk is, initially at rest, is dropped onto it. The second disk sticks in a position where is has a rotational inertia $9.75\,\mathrm{kg\,m^2}$ about the axle. Find the common final angular speed of the disks.



By conservation of angular momentum about the rotation axis,

$$L_f = L_i$$

$$(I_1 + I_2)\omega_f = I_1 \omega_i .$$

By the parallel axis theorem,

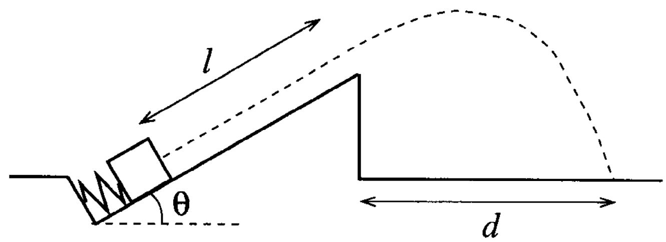
$$I_1 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

$$\omega_f = \frac{\tfrac{3}{2}MR^2}{\tfrac{3}{2}MR^2 + I_2}\,\omega_i$$

$$= \frac{\omega_i}{1 + 2I_2/(3MR^2)}$$

$$= 388 \text{ rev/min}$$

7. A spring having a force constant k is compressed a distance a from its natural length, then used to launch an ice cube of mass m up a ramp of length l oriented at an angle θ to the horizontal. The ice cube starts from rest at ground level, slides without friction up the ramp, and sails off the end. Find the horizontal distance d from the end of the ramp to the point where the ice cube returns to ground level. Neglect air resistance.



- Work done on the ice cube by ...

$$\left.\begin{array}{lll} \text{the spring} & & W_s = \frac{1}{2}ka^2 \\ \text{gravity} & & W_g = -mgl\sin\theta \end{array}\right\} \text{during motion to the top of the ramp}$$

Work-kinetic energy theorem:

$$\Delta K = W_{net}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}ka^2 - mgl\sin\theta$$

speed at top of ramp: $V_0 = \sqrt{\frac{ka^2}{m} - 2gl\sin\theta}$

- Trajectory through the air satisfies

$$y = y_0 + x\tan\phi_0 - \frac{gx^2}{2V_0^2\cos^2\phi_0}$$

with $y_0 = l\sin\theta$, $\phi_0 = \theta$, and V_0 as given above.

The landing point satisfies

$$0 = l\sin\theta + d\tan\theta - \frac{gd^2}{2V_0^2\cos^2\theta}$$

$$\Rightarrow \quad d = \frac{V_0^2\sin\theta\cos\theta}{g} + \frac{V_0\cos\theta}{g}\sqrt{V_0^2\sin^2\theta + 2gl\sin\theta}$$

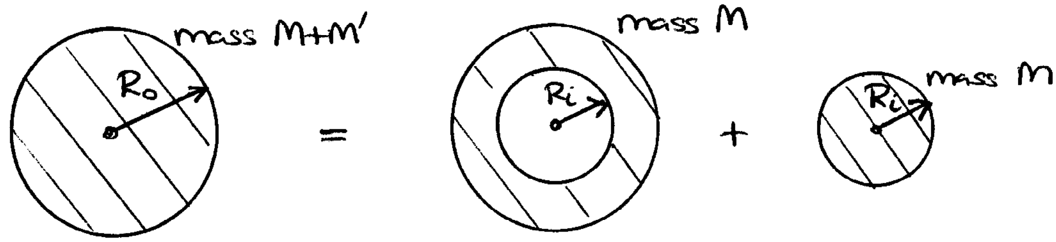
$$= \left(\frac{ka^2}{mg} - 2l\sin\theta\right)\sin\theta\cos\theta$$

$$+ \sqrt{\frac{ka^2}{m} - 2gl\sin\theta}\ \frac{\cos\theta}{g}\sqrt{\left(\frac{ka^2}{m} - 2gl\sin\theta\right)\sin\theta + 2gl\sin\theta}$$

8. A pipe of mass M has the shape of an annular cylinder with an inner radius R_i and an outer radius R_o. The walls of the pipe are of uniform density. Use the fact that the rotational inertia of a *uniform* cylinder of mass m and radius r about its long axis is $\frac{1}{2}mr^2$ to show that the rotational inertia of the pipe about its cylindrical axis is $\frac{1}{2}M(R_i^2 + R_o^2)$.

Hint: Recall the methods used to calculate the center-of-mass position of objects containing holes.

<u>Method 1</u>



$$\tfrac{1}{2}(M+M')R_o^2 = I + \tfrac{1}{2}M'R_i^2$$

$$I = \tfrac{1}{2}(M+M')R_o^2 - \tfrac{1}{2}M'R_i^2$$

$$= \tfrac{1}{2}MR_o^2 + \tfrac{1}{2}M'(R_o^2 - R_i^2)$$

Now $$\frac{M'}{M+M'} = \left(\frac{R_i}{R_o}\right)^2 \quad \text{(ratio of x-sectional areas)}$$

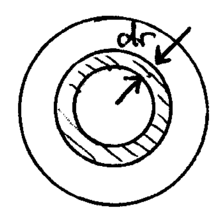
$$M' = \frac{(R_i/R_o)^2}{1-(R_i/R_o)^2}M$$

$$= \frac{R_i^2}{R_o^2 - R_i^2}M$$

$$\Rightarrow \quad I = \tfrac{1}{2}M(R_i^2 + R_o^2)$$

<u>Method 2</u>

$$I = \int r^2\,dm$$



$$= \rho L \int_{R_i}^{R_o} 2\pi r\,dr\, r^2$$

$$= \frac{\pi\rho L}{2}(R_o^4 - R_i^4)$$

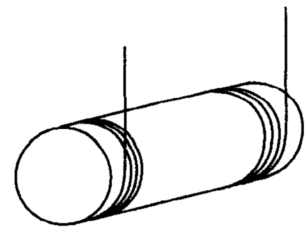
$$m = \int dm = \rho L \int_{R_i}^{R_o} 2\pi r\,dr$$

$$= \pi\rho L(R_o^2 - R_i^2)$$

$$\Rightarrow \quad I = \frac{M}{\pi\rho L(R_o^2 - R_i^2)}\frac{\pi\rho L}{2}(R_o^4 - R_i^4)$$

$$= \tfrac{1}{2}M(R_i^2 + R_o^2)$$

9. A solid, uniform cylinder has mass M, length L, and radius R. Two ideal cords are wrapped around the cylinder, one near each end, and the cord ends are firmly fixed to the ceiling. The cylinder is held stationary in a horizontal position, with the cords both taut and vertical, as shown in the figure. Then, the system is released from rest.

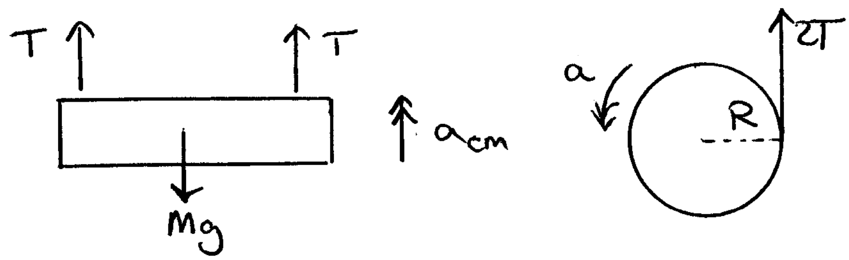


(a) Find the tension in each cord as it unwinds.

(b) Calculate the angular acceleration of the cylinder as it falls.

(c) Calculate the linear acceleration of the cylinder.

(a)



Applying N2 vertically

$$2T - Mg = Ma_{cm} \qquad (1)$$

Taking torques about the rotation axis of the cylinder

$$2TR = I_{cm}\alpha \qquad (2)$$

Here, $\quad I_{cm} = \frac{1}{2}MR^2$

and $\quad \alpha = -\frac{a_{cm}}{R} \quad$ (note sign)

(2) $\Rightarrow \quad 2TR = -\frac{1}{2}MRa_{cm} \qquad (3)$

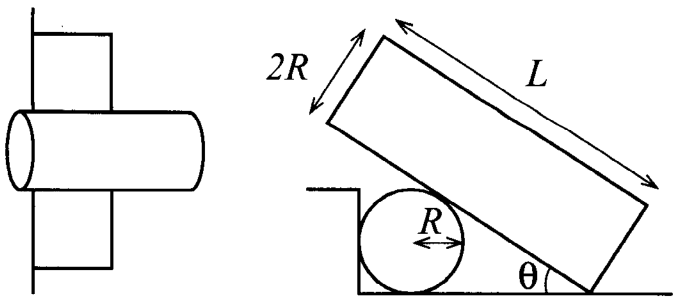
(1), (3) $\Rightarrow \quad T = \frac{1}{6}Mg$

(b) Substituting for T in (2),

$$\alpha = \frac{2g}{3R}$$

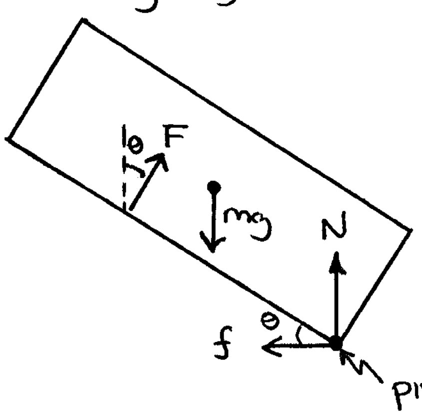
(c) Finally $\quad a = -\frac{2g}{3} \quad$ (downward)

10. A solid, uniform cylinder of length L and radius R rests on a level floor. The cylinder is in contact all along its length with the vertical face of a step. A second, identical cylinder rests both on the ground (at one end) and on the first cylinder, as shown in the two views below. The long axis of the second cylinder makes an angle θ with the ground. Friction between the two cylinders is negligible. Find the minimum coefficient of static friction between the cylinders and the ground that allows static equilibrium to be maintained.

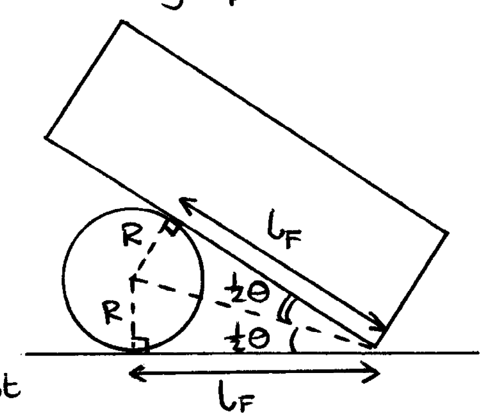


top view side view

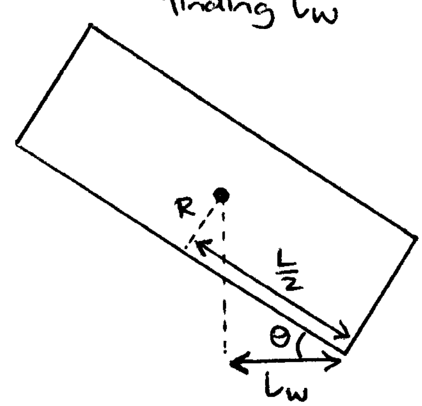
free-body diagram



finding l_F



finding l_w



Equilibrium equations for the upper cylinder:

$$0 = \Sigma F_x = F\sin\theta - f \qquad (1)$$

$$0 = \Sigma F_y = F\cos\theta + N - mg \qquad (2)$$

$$0 = \Sigma \tau_z = mg\,l_w - F\,l_F \qquad (3)$$

moment arm for force F is $\quad l_F = R\cot\theta/2 \qquad (4)$

" " " " mg is $\quad l_w = \frac{L}{2}\cos\theta - R\sin\theta \qquad (5)$

At the limit of static friction, $f = \mu_s N \qquad (6)$

$$(1),(6) \Rightarrow \mu_{s,min} = \frac{F\sin\theta}{N} \overset{\times l_f}{\underset{(2)}{=}} \frac{(Fl_f)\sin\theta}{mgl_f - (Fl_f)\cos\theta} \overset{(3)}{=} \frac{mgl_w\sin\theta}{mg\,l_f - mgl_w\cos\theta}$$

$$\overset{(4)(5)}{=} \frac{(\frac{L}{2}\cos\theta - R\sin\theta)\sin\theta}{R\cot\frac{\theta}{2} - (\frac{L}{2}\cos\theta - R\sin\theta)\cos\theta}$$