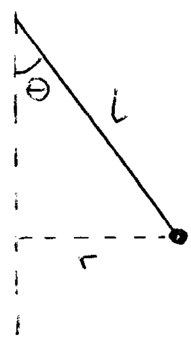
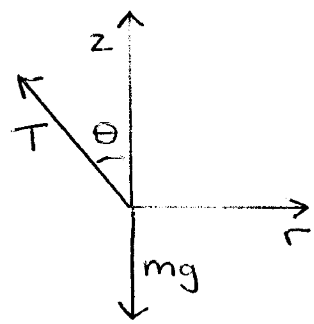
PHY 2060 Spring 2007 - Exam 2

1. (a) A conical pendulum consists of a mass m attached to an ideal string of length l, the other end of which is anchored to a fixed point. When the pendulum is set in steady motion, the mass rotates at a constant speed v in a horizontal, circular orbit of radius r. Assume that air resistance is negligible.

   Place a check to the left of any/all of the following statements that is/are necessarily true:

   i. The tension in the string has magnitude mv^2/r.
   ii. The tension in the string is constant in time.
   iii. The vertical forces acting on the mass sum to zero.
   iv. The horizontal forces acting on the mass sum to zero.
   v. A centrifugal force of magnitude mv^2/r acts on the mass to stop it hanging directly underneath the string's anchor point.





N2 along z: $T\cos\theta - mg = 0$

" " r: $-T\sin\theta = -\frac{mv^2}{r}$

$\Big\}\ T = m\sqrt{g^2 + v^4/r^2}$

(i) $T > mv^2/r$, so FALSE.

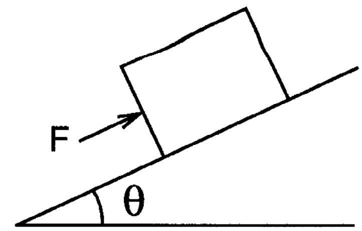
(ii) $T =$ constant, but the direction changes. TRUE and FALSE are both acceptable.

(iii) TRUE. This is necessary for equilibrium.

(iv) FALSE. The horizontal forces must sum to the centripetal force.

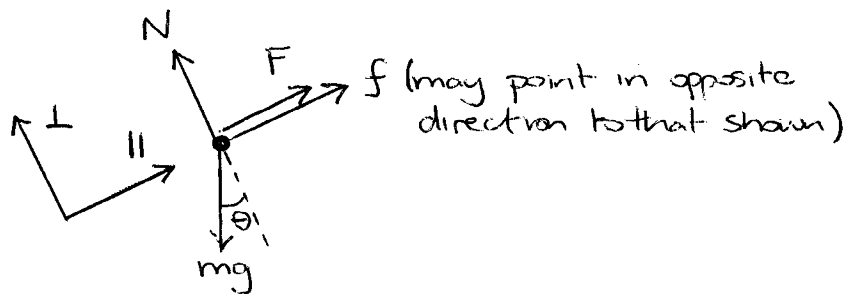
(v) FALSE. There is no such thing as a centrifugal force.

(b) A box of mass m is on an incline that makes an angle θ to the horizontal. The coefficient of static friction between the box and the slope is $\mu_s > 0$. The box remains stationary while you apply a force of magnitude F to the box as shown in the figure. This force is directed parallel to the slope.



Place a check to the left of any/all of the following statements that is/are necessarily true:

i. The applied force satisfies $F \leq mg(\sin\theta + \mu_s \cos\theta)$.
ii. The applied force satisfies $F \leq mg(\sin\theta - \mu_s \cos\theta)$.
iii. The applied force is $F = mg(\sin\theta - \mu_s \cos\theta)$.
iv. The applied force satisfies $F \geq mg \sin\theta$.
v. The applied force satisfies $F \leq mg \sin\theta$.
vi. The applied force is $F = mg \sin\theta$.



N2 along $\perp$: $\quad N - mg\cos\theta = 0 \Rightarrow N = mg\cos\theta$

" " $\parallel$: $\quad F + f - mg\sin\theta = 0 \Rightarrow F = mg\sin\theta - f$

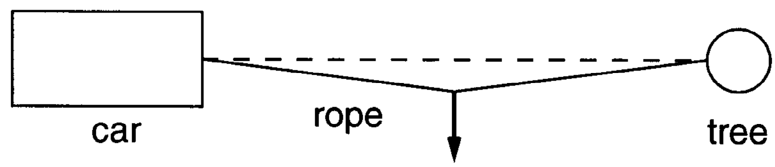
Static friction: $\quad |f| \leq \mu_s N = \mu_s mg\cos\theta$

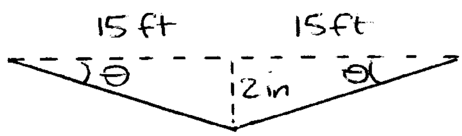
Thus,

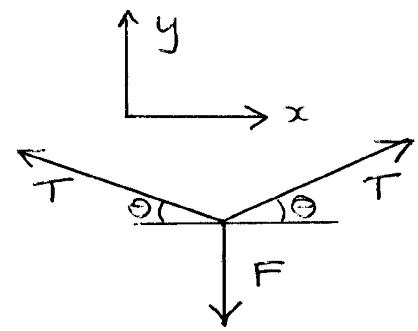
$$mg(\sin\theta - \mu_s\cos\theta) \leq F \leq mg(\sin\theta + \mu_s\cos\theta)$$

Only statement (i) is TRUE.

2. Your car gets stuck in mud. To try to free the car, you attach one end of a rope to the car and tie the rope tautly around an oak tree 30 ft away. You then pull sideways at the midpoint of the rope, as shown in the figure. Find the force exerted on the car when you pull on the rope with a 500-N force and the midpoint of the rope is displaced by 2 in from its original position. (Assume that the car does not move.)







$$\tan\theta = \frac{2\text{ in}}{15\text{ ft}} = \frac{2}{15\times 12}$$

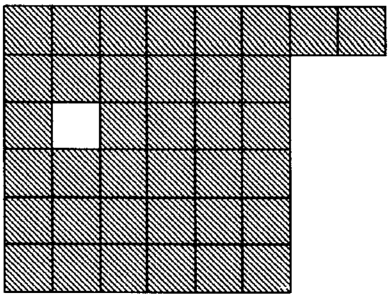
$$\theta \simeq 0.64^\circ$$

N2 along y: $2T\sin\theta - F = 0$

$$T = \tfrac{1}{2}F\csc\theta$$

$$= 2.3\times 10^{4}\text{ N}$$

3. A plate, of uniform thickness and uniform density, is in the shape of a 6×6 square with a 1×1 hole and a 2×1 protrusion, as shown in the figure. Find the center of mass position relative to the bottom left corner of the plate.



The cm position of a composite body satisfies

$$\vec{r}_{cm} = \frac{\sum_j m_j \vec{r}_{cm,j}}{\sum_j m_j}$$

Break the plate above into a 6×6 square, a 2×1 rectangle, and a 1×1 square of negative mass:

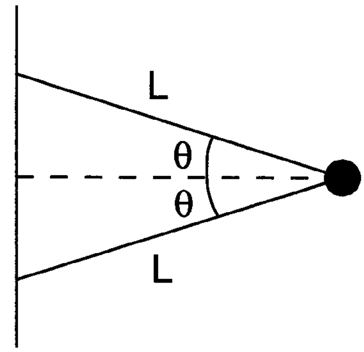
$$x_{cm} = \frac{36 \times 3 + 2 \times 7 - 1 \times 1\tfrac{1}{2}}{36 + 2 - 1}$$

$$= \frac{241}{74} \simeq 3.26$$

$$y_{cm} = \frac{36 \times 3 + 2 \times 5\tfrac{1}{2} - 1 \times 3\tfrac{1}{2}}{36 + 2 - 1}$$

$$= \frac{231}{74} \simeq 3.12$$

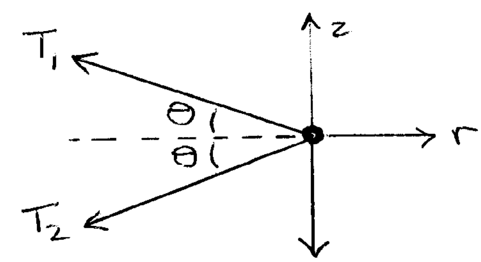
4. A point mass m is attached to two ideal strings, each of length L. The strings are attached to a vertical rod, and the rod is rotated in such a manner that the mass undergoes uniform circular motion at speed v in a horizontal plane, and the strings make equal angles θ to the horizontal (as shown in side view at right).



(a) Find the tension in each of the strings.

(b) What is the minimum value of v for which the configuration shown is stable?

(a) Free-body diagram:



N2 along z: $$(T_1 - T_2)\sin\theta - mg = 0 \quad (1)$$

" " r: $$-(T_1 + T_2)\cos\theta = -\frac{mv^2}{r} = -\frac{mv^2}{L}\sec\theta \quad (2)$$

(1) $\Rightarrow$ $$T_1 - T_2 = mg\csc\theta$$

(2) $\Rightarrow$ $$T_1 + T_2 = \frac{mv^2}{L}\sec^2\theta$$

$$T_1 = \frac{m}{2}\left(\frac{v^2}{L}\sec^2\theta + g\csc\theta\right)$$

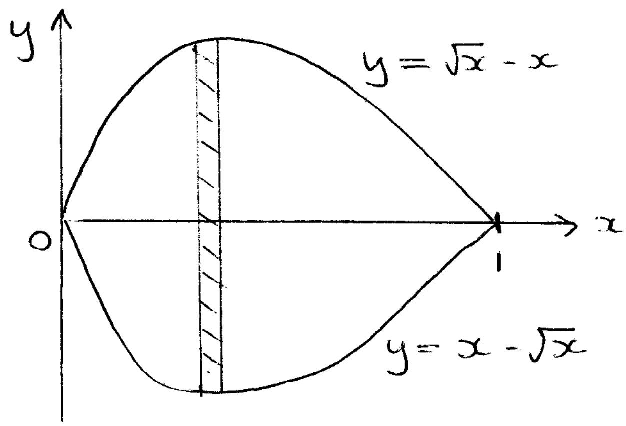
$$T_2 = \frac{m}{2}\left(\frac{v^2}{L}\sec^2\theta - g\csc\theta\right)$$

(b) The configuration is stable so long as both strings are taut, i.e., $T_1 \geq 0$ and $T_2 \geq 0$.

This requires

$$v \geq \sqrt{gL\cos\theta\cot\theta}$$

5. A two-dimensional object is formed by the region of the xy plane enclosed by the curves $y = \sqrt{x} - x$ and $y = x - \sqrt{x}$. Find the coordinates of the object's center of mass.



By symmetry, $\quad y_{cm} = 0$

while $\quad x_{cm} = \dfrac{\int x\, dm}{\int dm} = \dfrac{\int x\, dA}{\int dA}$

where dm and dA are mass and area elements, respectively.

The area element is the shaded strip above:

$$dA = (\text{height})\, dx$$
$$= 2(\sqrt{x} - x)\, dx$$

$$x_{cm} = \frac{\int_0^1 x \; \cancel{2}(\sqrt{x} - x)\, dx}{\int_0^1 \cancel{2}(\sqrt{x} - x)\, dx}$$

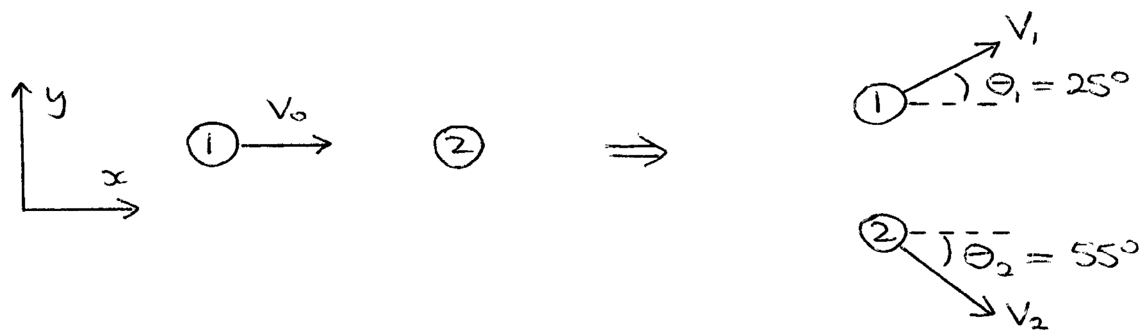
$$= \frac{\int_0^1 (x^{3/2} - x^2)\, dx}{\int_0^1 (x^{1/2} - x)\, dx}$$

$$= \frac{\left[\frac{2}{5} x^{5/2} - \frac{1}{3} x^3\right]_0^1}{\left[\frac{2}{3} x^{3/2} - \frac{1}{2} x^2\right]_0^1}$$

$$= \left(\frac{6}{15} - \frac{5}{15}\right) / \left(\frac{4}{6} - \frac{3}{6}\right)$$

$$= \frac{2}{5}$$

6. A hockey puck of mass 0.15 kg moving at 3.0 m/s along the $+x$ axis slides into a second, identical puck that is initially stationary. After the collision, the first puck is traveling at speed v_1 at angle $+25°$ measured (counter-clockwise) from the $+x$ axis; the second puck is traveling at speed v_2 at angle $-55°$ measured from the $+x$ axis. Find v_1 and v_2.



Conservation of linear momentum

along x: $\quad mV_o + 0 = mV_1\cos\theta_1 + mV_2\cos\theta_2 \quad (1)$

along y: $\quad 0 + 0 = mV_1\sin\theta_1 - mV_2\sin\theta_2 \quad (2)$

$(2) \Rightarrow \quad V_2 = V_1 \dfrac{\sin\theta_1}{\sin\theta_2}$

$(1) \Rightarrow$

$$V_o = V_1\left(\cos\theta_1 + \frac{\sin\theta_1}{\sin\theta_2}\cos\theta_2\right)$$

$$= \frac{V_1}{\sin\theta_2}\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)$$

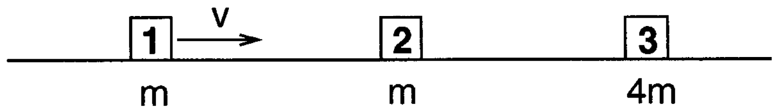
$$= \frac{V_1\sin(\theta_1+\theta_2)}{\sin\theta_2}$$

$$V_1 = \frac{V_o\sin\theta_2}{\sin(\theta_1+\theta_2)} = 2.50 \text{ m/s}$$

$$V_2 = \frac{V_o\sin\theta_1}{\sin(\theta_1+\theta_2)} = 1.29 \text{ m/s}$$

Note that this is not an elastic collision. We learned in homework that for the elastic case, $\theta_1 + \theta_2 = 90°$.

7. Three carts are spaced out as shown in the figure along a straight track that permits the carts to move (without friction) in only one dimension. Carts 1 and 2 have mass m, and cart 3 has mass $4m$. Carts 2 and 3 are initially stationary, while cart 1 is initially moving towards cart 2 at speed v. Assume that all subsequent collisions are elastic. Find the final velocity of each cart after all collisions are over.



For an elastic collision in 1D,

$$V_{Af} = \frac{m_A - m_B}{m_A + m_B} V_{Ai} + \frac{2m_B}{m_A + m_B} V_{Bi}$$

$$V_{Bf} = \frac{m_B - m_A}{m_B + m_A} V_{Bi} + \frac{2m_A}{m_B + m_A} V_{Ai}$$

After the first collision, between 1 and 2,

$V_1 = 0$, $\quad V_2 = V$, $\quad V_3 = 0$.

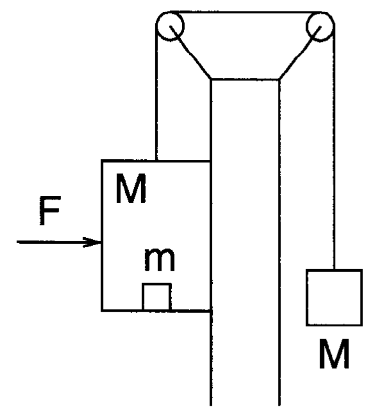
After the second collision, between 2 and 3,

$V_1 = 0$, $\quad V_2 = -\frac{3}{5}V$, $\quad V_3 = \frac{2}{5}V$.

After the third and final collision, between 1 and 2,

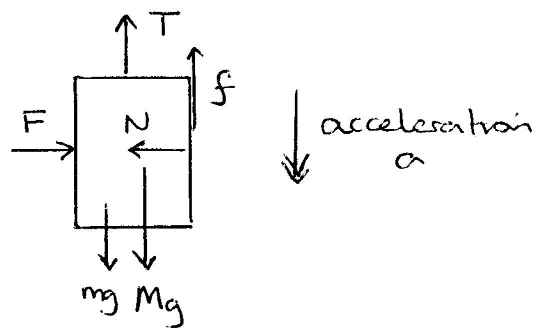
$V_1 = -\frac{3}{5}V$, $\quad V_2 = 0$, $\quad V_3 = \frac{2}{5}V$.

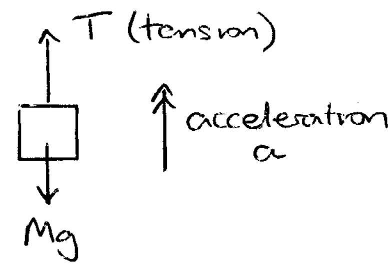
8. Consider a crude elevator formed from a cage of mass M hanging from an ideal string that passes over a pair of ideal pulleys and is attached at the other end to a counterweight, which also has mass M. When a load m is placed in the cage, it begins to accelerate downward. For safety reasons, it is necessary to limit the speed with which the elevator descends. This goal can be accomplished by pressing the side of the cage against a vertical wall with a horizontal force F. The coefficient of kinetic friction between the cage and the wall is μ_k.



(a) Assume initially that $F = 0$. What is the magnitude of the elevator's acceleration and what is the tension in the string ?

(b) Now let F be nonzero. What value must it take to keep the elevator descending at a constant velocity?





N2 applied to....

the cage: $\quad T - (M+m)g + f = -(M+m)a$

the counterweight: $\quad T - Mg = Ma$

Kinetic friction: $\quad f = \mu_k N = \mu_k F$

(a) $f = 0 \Rightarrow$

$$a = \frac{m}{2M+m} g$$

$$T = \frac{2(M+m)}{2M+m} Mg \quad (> Mg)$$

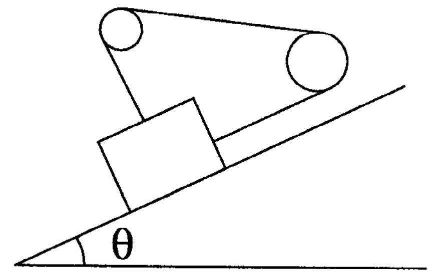
(b) $a = 0 \Rightarrow$

$$T = Mg$$

$$f = mg$$

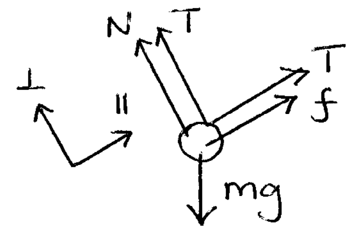
$$F = \frac{mg}{\mu_k}$$

9. A block of mass m is in contact with a plane inclined at angle θ to the horizontal. The coefficient of static friction between the block and the slope μ_s is too small for the block to remain in equilibrium on the slope by itself. In order to maintain equilibrium, an ideal string is attached to the block in two places, as shown in the figure. The string is tensioned by passing it over two ideal pulleys, which are positioned so that the string runs parallel to the slope at one point of attachment to the block and perpendicular to the slope at the other attachment point.



(a) Find the greatest and smallest values of the tension in the string for which the system can remain in static equilibrium, assuming that $\mu_s < 1$.

(b) Repeat part (a), assuming instead that $\mu_s > 1$.



N2 $\perp$: $T + N - mg\cos\theta = 0$ ①

N2 $\parallel$: $T + f - mg\sin\theta = 0$ ②

Friction: $|f| \leq \mu_s N$ ③

①,②,③ $\Rightarrow$ $-\mu_s(mg\cos\theta - T) \leq mg\sin\theta - T \leq \mu_s(mg\cos\theta - T)$ ④

- Without the string ($T=0$) equilibrium requires $\mu_s \geq \tan\theta$. The question states that equilibrium is not maintained, so

$$\mu_s < \tan\theta \quad \text{and} \quad T > 0. \qquad ⑤$$

- The contact between block and plane is stable so long as $N \geq 0$, so ① $\Rightarrow$

$$T \leq mg\cos\theta \qquad ⑥$$

- ④ $\Rightarrow$

$$T_{min} = mg\cos\theta \, \frac{\tan\theta - \mu_s}{1 - \mu_s} \qquad ⑦$$

$$T_{max} = mg\cos\theta \, \frac{\tan\theta + \mu_s}{1 + \mu_s} \qquad ⑧$$

(a) For $\mu_s < 1$, ⑤–⑧ are all consistent provided $\theta \leq 45°$.
If $\theta > 45°$, ⑥ and ⑦ are inconsistent so there is no stable equilibrium possible.

(b) For $\mu_s > 1$, making $T > 0$ decreases f by an amount greater than T, making it impossible to satisfy ②, so there is no stable equilibrium possible.

10. A 70-kg stuntman takes off straight upwards from the ground wearing a jet pack that burns 2.0 kg of fuel per second and ejects the burned fuel vertically downward at an exhaust speed of 750 m/s (relative to the pack). The initial mass of fuel is 50 kg, while the non-fuel parts of the jet pack have a mass of 20 kg. Find the stuntman's velocity at the moment the fuel runs out. Neglect air resistance and any change of the gravitational constant g with altitude.

Hint: Start from the rocket equation

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{\text{ext}} + \mathbf{v}_{\text{rel}}\frac{dM}{dt}.$$

The external force has magnitude Mg, where M is the combined mass of the stuntman, the jet pack, and the remaining fuel. You should be able to integrate this differential equation almost as easily as in the case $\mathbf{F}_{\text{ext}} = \mathbf{0}$ treated in class.

Resolving the equation of motion along the $+y$ (vertical) direction,

$$M\frac{dv}{dt} = -Mg - V_{rel}\frac{dM}{dt}$$

where $V_{rel} > 0$ and $\frac{dM}{dt} < 0$.

$$\Rightarrow \quad dv = -g\,dt - V_{rel}\frac{dM}{M}$$

$$\int_{V_i}^{V_f} dv = -g\int_{t_i}^{t_f} dt - V_{rel}\int_{M_i}^{M_f}\frac{dM}{M}$$

$$V_f - V_i = -g(t_f - t_i) - V_{rel}\ln\frac{M_f}{M_i}$$

$$V_f = V_i - g(t_f - t_i) + V_{rel}\ln\frac{M_i}{M_f}$$

Here

$$V_i = 0$$

$$M_i = 140\text{kg}$$

$$M_f = 90\text{ kg}$$

$$t_f - t_i = \frac{M_f - M_i}{dM/dt} = \frac{-50\text{kg}}{-2\text{ kg/s}} = 25\text{s}$$

$$\Rightarrow \quad V_f = 81.4\text{ m/s}.$$

Note: An alternative is to write $M(t) = M_i - \alpha t$ where $\alpha = -dM/dt = $ constant, and then integrate

$$dv = \left(-g + \frac{V_{rel}\,\alpha}{M_0 - \alpha t}\right)dt$$