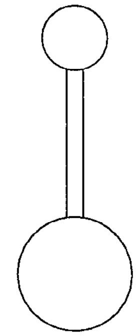
PHY 2060 SPRING 2007 - EXAM 3

1. (a) A dog's toy consists of two uniform rubber spheres connected by a uniform rubber cylinder, whose axis lies along the line between the balls' centers (see figure). The cylinder's length is much greater than its diameter. The spheres have different diameters.



Let A be a rotation axis along the axis of the cylinder, B be a rotation axis perpendicular to the axis of the cylinder passing through the cylinder's center of mass, and C be a rotation axis parallel to B passing through the center of one of the spheres. Let I_X denote the rotational inertia of the toy about axis X.

Place a check to the left of any/all of the following statements that is/are **necessarily** true:

× i. The rotational inertias satisfy $I_A > I_B$.

✓ ii. The rotational inertias satisfy $I_B > I_A$.

✓ iii. The rotational inertias satisfy $I_C > I_A$.

× iv. The rotational inertias satisfy $I_C > I_B$.

× v. The rotational inertias satisfy $I_B > I_C$.

$$I_A = \underbrace{\tfrac{2}{5} M_1 R_1^2}_{\text{sphere 1}} + \underbrace{\tfrac{2}{5} M_2 R_2^2}_{\text{sphere 2}} + \underbrace{\tfrac{1}{2} M_c R_c^2}_{\text{cylinder}}$$

$$I_B = \tfrac{2}{5} M_1 R_1^2 + M_1\left(\tfrac{L}{2} + R_1\right)^2$$
$$+ \tfrac{2}{5} M_2 R_2^2 + M_2\left(\tfrac{L}{2} + R_2\right)^2$$
$$+ \underbrace{\tfrac{1}{12} M_c L^2 + \tfrac{1}{4} M_c R_c^2}_{\text{Since } L \gg R_c\text{, this is bigger than } \frac{1}{2} M_c R_c^2}$$

parallel-axes theorem

$$> I_A$$

$$I_C = \tfrac{2}{5} M_1 R_1^2 + \tfrac{2}{5} M_2 R_2^2 + M_2 (L + R_1 + R_2)^2$$
$$+ \tfrac{1}{12} M_c L^2 + M_c\left(\tfrac{L}{2} + R_1\right)^2$$

(assuming axis C passes through sphere 1)

$$> I_A$$

$$\gtrless I_B \text{ depending on which sphere is larger}$$

(b) A particle of mass m has Cartesian coordinates (measured in meters) $x = -2t^2$, $y = 3$, $z = 0$, where $t \geq 0$ is the time (measured in seconds). The statements below concern the particle's angular momentum $\mathbf{l}$ about the origin $x = y = z = 0$.

Place a check to the left of any/all of the statements that is/are **necessarily** true:

✓ i. The angular momentum $\mathbf{l}$ has a magnitude that increases continuously from time $t = 0$.

× ii. The angular momentum $\mathbf{l}$ has a magnitude that decreases continuously from time $t = 0$.

× iii. The angular momentum $\mathbf{l}$ has a constant magnitude.

× iv. The angular momentum $\mathbf{l}$ points along the negative z axis.

× v. The angular momentum $\mathbf{l}$ points along the negative x axis.

Particle's position at time t is

$$\vec{r}(t) = -2t^2\,\hat{\imath} + 3\hat{\jmath}$$

Velocity $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt} = -4t\,\hat{\imath}$

Momentum $\qquad \vec{p}(t) = m\vec{v}(t) = -4mt\,\hat{\imath}$

Angular momentum $\qquad \vec{l}(t) = \vec{r}(t) \times \vec{p}(t)$

$$= (-2t^2\hat{\imath} + 3\hat{\jmath}) \times (-4mt\,\hat{\imath})$$

$$= +12mt\,\hat{k}$$

(points along +z axis)

2. A car has wheels of diameter 70 cm. During the time that the car brakes to a complete stop from an initial speed of 45 km/h, the wheels complete 25 revolutions.

   (a) What is the initial angular speed of the wheels? Give your answer in rad/s.

   (b) What is the average angular acceleration of the wheels during the time the car is braking?

(a) Assuming rolling without slipping,

$$V_{car} = r\omega$$

where $r = \frac{1}{2}(\text{diameter}) = 35\,\text{cm}$

Initially,

$$\omega_i = \frac{V_i}{r}$$

$$= \frac{45\,\text{km/h}}{35\,\text{cm}}\left(\frac{1000\,\text{m}}{1\,\text{km}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right)\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)$$

$$= 36\,\text{rad/s}$$

(b) Constant angular acceleration:

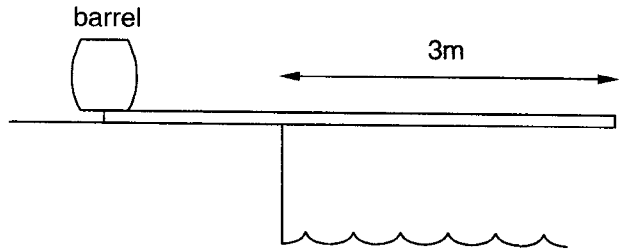
$$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\phi$$

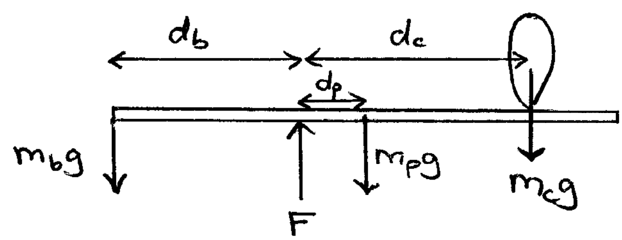
$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\phi}$$

$$= \frac{0 - (35.7\,\text{rad/s})^2}{2\,(25\,\text{rev})\,(2\pi\,\text{rad/rev})}$$

$$= -4.1\,\text{rad/s}^2$$

3. The crew of a pirate ship mutiny and vote to make their former captain walk the plank. A straight, uniform plank of mass 25 kg and length 5 m is placed on the flat deck of the ship so that 3 m of the plank extend overboard, as shown in the figure. A 100-kg barrel of water is placed on the plank at its inboard end to help anchor the plank in place. If the deposed captain has a mass of 80 kg, how close to the seaward end of the plank can he advance before he and the plank topple into the water?





At the point where the plank becomes unstable, the force of the deck acts only at the edge, which is the (potential) pivot point.

Taking torques about this pivot, stability requires

$$m_b g d_b - m_p g d_p - m_c g d_c = 0$$

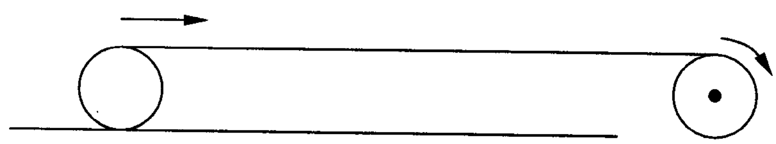
$$d_c = \frac{m_b d_b - m_p d_p}{m_c}$$

$$= \frac{(100\,\text{kg})(2\,\text{m}) - (25\,\text{kg})(0.5\,\text{m})}{80\,\text{kg}}$$

$$= 2.34\,\text{m}$$

At the moment he begins to topple, the captain is 2.34 m from the pivot or 0.66 m from the seaward end of the plank.

4. A cylinder of radius 30 cm rests on a horizontal surface. An ideal string is partially wrapped around the cylinder, and the other end is wrapped around the rim of a wheel, also of radius 30 cm. The long segment of string between the cylinder and the wheel is taut and horizontal. At time $t = 0$, a drive motor begins to rotate the wheel with a constant angular acceleration of $0.2\,\mathrm{rev/s^2}$. This rotation winds the string onto the wheel, and causes the cylinder to roll without slipping across the surface towards the wheel. How long does it take for the cylinder's center of mass to travel the first 3.5 m?



Angle turned by wheel after time t is

$$\Delta\phi = \cancel{\omega_0 t} + \tfrac{1}{2}\alpha t^2$$

Length of string wound onto wheel is

$$L = r\Delta\phi = \tfrac{1}{2}\alpha r t^2.$$

In rolling without slipping, the point of attachment at the top of the cylinder moves twice as fast as the CM.

$\Rightarrow$ CM distance traveled after time t is

$$S_{cm} = \tfrac{1}{2}L = \tfrac{1}{4}\alpha r t^2.$$

$$t = \sqrt{\frac{4 S_{cm}}{\alpha r}}$$

$$= \sqrt{\frac{4 \times (3.5\,\mathrm{m})}{(0.2\,\mathrm{rev/s^2})(2\pi\,\mathrm{rad/rev})(0.30\,\mathrm{m})}}$$

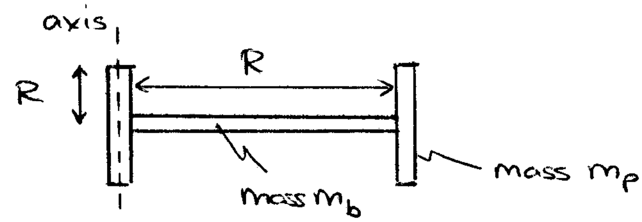
$$= 6.1\,\mathrm{s}$$

5. A barbell consists of a 2-m long, thin, uniform bar of mass 20 kg and two identical, circular plates whose thickness is negligible compared to their 40-cm radius. Each end of the bar is welded to the center of one of the plates, so that the bar is perpendicular to the plane of the plate. The rotational inertia of this barbell, when rotated about an axis perpendicular to the barbell passing through the center of one of the plates, is $231\,\mathrm{kg\,m^2}$.

   (a) Find the mass of each of the plates.

   (b) Calculate the rotational inertia of the barbell about an axis perpendicular to the bar passing through the bar's center.

(a)



Neglecting the thicknesses of the bar and the plates, the rotational inertia about the axis shown is

$$I = \tfrac{1}{4}m_p R^2 + \left(\tfrac{1}{4}m_p R^2 + m_p L^2\right) + \tfrac{1}{3}m_b L^2$$

$$= m_p\left(\tfrac{1}{2}R^2 + L^2\right) + \tfrac{1}{3}m_b L^2$$

$$\Rightarrow \quad m_p = \frac{I - \frac{1}{3}m_b L^2}{\frac{1}{2}R^2 + L^2}$$

$$= \frac{231 - \frac{1}{3}(20)(2)^2}{\frac{1}{2}(0.4)^2 + (2)^2}\ \mathrm{kg}$$

$$= 50\ \mathrm{kg}$$

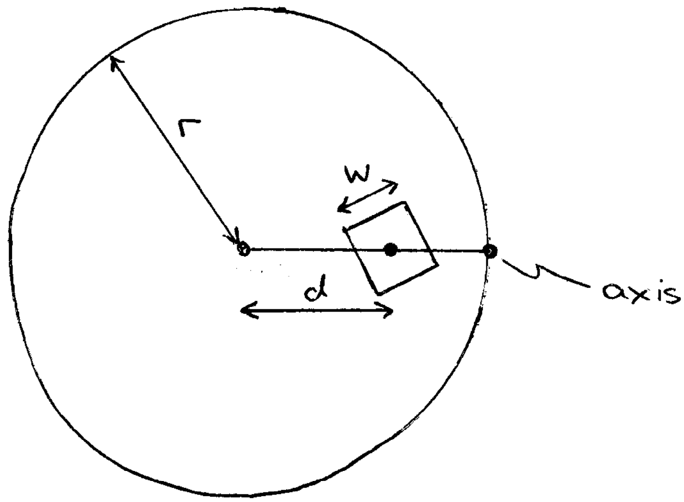
(b) The rotational inertia about an axis through the CM is

$$I_{cm} = 2\left[\tfrac{1}{4}m_p R^2 + m_p\left(\tfrac{L}{2}\right)^2\right] + \tfrac{1}{12}m_b L^2$$

$$= 2\left[\tfrac{1}{4}(50)(0.4)^2 + (50)(1)^2\right] + \tfrac{1}{12}(20)(2)^2\ \mathrm{kg\,m^2}$$

$$= 111\ \mathrm{kg\,m^2}$$

6. A thin, circular plate has radius r, uniform thickness t, and uniform density ρ. A square hole, with sides of length w, is cut from the plate. The square lies entirely inside the circular perimeter, with its center located a distance d from the center of the circle. Find the rotational inertia of the plate about an axis perpendicular to the plane of the plate, passing through the point on the outer rim of the plate closest to the center of the hole.



Use the "negative-mass" trick:

$$I_{plate} = I_{\substack{complete\\circle}} - I_{\substack{filled\\hole}}$$

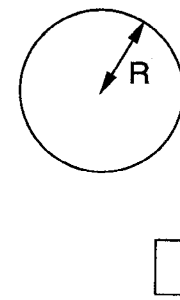
$$= \left(\tfrac{1}{2}m_c r^2 + m_c r^2\right) - \left(\tfrac{1}{6}m_h w^2 + m_h (r-d)^2\right)$$

using the parallel-axes theorem for each object separately.

Since
$$m_c = \pi r^2 t \rho$$
$$m_h = w^2 t \rho$$

$$\Rightarrow \quad I_{plate} = \left[\frac{3\pi r^4}{2} - \frac{w^4}{6} - (r-d)^2 w^2\right] t\rho$$

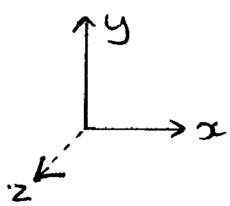
7. A solid, uniform sphere of mass M and radius R pivots on a fixed, massless, frictionless axle that passes through the center of the sphere. An ideal string is partially wrapped around the sphere's equator so that the sphere rotates when a mass $M/2$ is hung from the other end of the string.

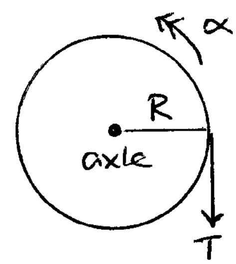


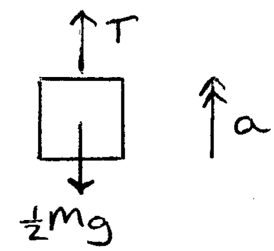
(a) What is the mass' acceleration just after this system is released from rest?

(b) What is the angular momentum of the system about the axle at time t after the system was released from rest? (Assume that at time t, the string has not yet fully unwound from the sphere.)

(a)







About axle, along z axis: $\sum_i \tau_i = I\alpha$

$$-TR = \frac{2}{5}MR^2\alpha \quad (1)$$

Along y: $\sum_i F_i = ma$

$$T - \frac{1}{2}Mg = \frac{1}{2}Ma \quad (2)$$

Kinematic constraint: $a = R\alpha \quad (3)$

Solution to (1) - (3) is $a = -\frac{5}{9}g$

Note: $T = \frac{1}{2}Mg$ only if $a = 0$.

$|a| < g$ because taut string retards the mass' fall.

(b) Total ang. momentum

$$\vec{L} = \underset{\text{sphere}}{I\vec{\omega}} + \underset{\text{mass } \frac{1}{2}M}{\vec{r} \times m\vec{v}}$$

$$= L_z\hat{k} \text{ here}$$

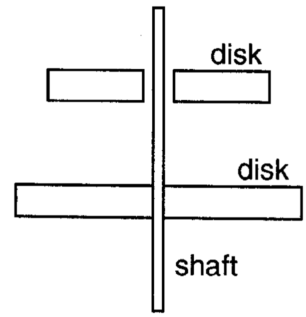
Since

$$v_y = R\omega = -\frac{5gt}{9}$$

$$L_z = \frac{2}{5}MR^2\left(-\frac{5gt}{9R}\right) + R\frac{M}{2}\left(-\frac{5gt}{9}\right)$$

$$= -\frac{1}{2}MRgt$$

8. The lower disk in the figure has mass 240 g and radius 3.5 cm. It is initially rotating at 80 rev/min on a light, frictionless shaft of negligible radius. The upper disk, of mass 110 g and radius 3.0 cm, is at rest. It is allowed to drop freely down the shaft onto the lower disk, and frictional forces act to bring the two disks to a common angular velocity. The disks are of uniform thickness and density.



   (a) What is the common angular velocity of the two disks?

   (b) What is the magnitude of the constant torque (measured about the shaft) that would be required to bring the two disks to rest in a time of 20 s?

(a) Total angular momentum about the shaft is conserved:

$$(I_U + I_L)\,\omega_f = I_L \omega_i$$

$$\omega_f = \frac{I_L}{I_U + I_L}\,\omega_i = \frac{\frac{1}{2}m_L R_L^2 \omega_i}{\frac{1}{2}m_U R_U^2 + \frac{1}{2}m_L R_L^2}$$

$$= \frac{240\times(3.5)^2}{110\times(3.0)^2 + 240\times(3.5)^2}\; 80\text{ rev/min}$$

$$\simeq 60\text{ rev/min or } 6.26\text{ rad/s}$$

(b)

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\Rightarrow \quad |\tau_{avg}| = \left|\frac{\Delta L}{\Delta t}\right|$$

$$= \frac{I_L \omega_i}{\Delta t} = \frac{m_L R_L^2 \omega_i}{2\Delta t}$$

$$= \frac{(0.24\text{ kg})(0.035\text{m})^2(80\text{ rev/min})}{2\,(20\text{s})}$$

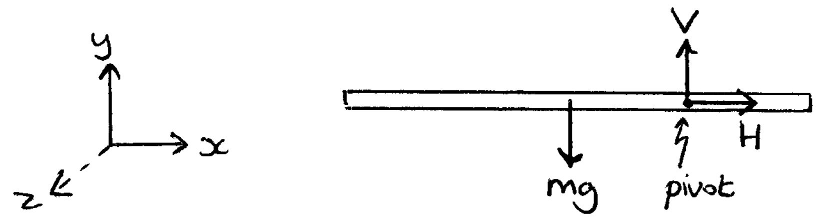
$$\times \left(\frac{2\pi\text{ rad}}{1\text{rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)$$

$$\simeq 6.2\times 10^{-5}\text{ Nm}$$

9. A uniform meter stick can rotate freely about an axle passing through a hole bored at the 75-cm mark. The stick is held in a horizontal position and then released.

(a) What is the stick's angular acceleration immediately after it is released?

(b) What is the magnitude and direction of the force exerted by the axle on the meter stick immediately after the stick is released?



(a) Taking torques along z about the axle:

$$\Sigma \tau_z = I \alpha_z$$

$$mg \frac{L}{4} = \left[\frac{1}{12} m L^2 + m\left(\frac{L}{4}\right)^2\right] \alpha_z$$

$$\alpha_z = \frac{12g}{7L}$$

$$\simeq 17 \text{ rad/s}^2$$

(b) N2 along x: $H = m a_{cm,x}$

" " y: $V - mg = m a_{cm,y}$

Kinematic constraints (rigid-body rotation):

$$a_{cm,x} = 0$$

$$a_{cm,y} = -\frac{L}{4}\alpha_z$$

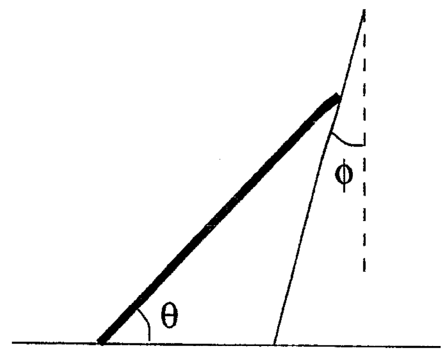
$\Rightarrow$ The components of the force exerted by the axle are

$$H = 0$$

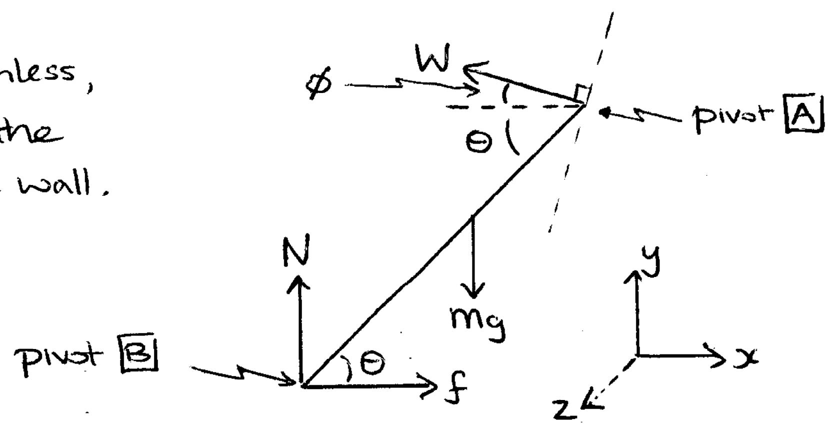
$$V = \frac{4}{7} mg$$

The force exerted by the axle on the meterstick equals $\frac{4}{7}$ of the stick's weight, directed vertically upward.

10. A ladder has mass M, length L, and can be treated as having a uniform mass per unit length. The ladder leans against a flat, frictionless wall that makes an angle ϕ with the vertical as shown in the figure. The coefficient of static friction between the ladder and the horizontal ground is μ_s. What is the minimum angle θ between the ladder and the ground that will ensure that the ladder remains stable? You should aim to get a final result of the form $\theta = \text{atan}(\ldots)$, where "..." does not depend on θ.



Since the wall is frictionless, the force it exerts on the ladder is normal to the wall.



$$\sum F_x = 0 \Rightarrow \qquad f - W\cos\phi = 0 \qquad (1)$$

$$\sum F_y = 0 \Rightarrow \qquad N + W\sin\phi - Mg = 0 \qquad (2)$$

$$\sum \tau_z = 0 \Rightarrow fL\sin\theta - NL\cos\theta + Mg\tfrac{L}{2}\cos\theta = 0 \qquad (3) \quad \text{about } \boxed{A}$$

$$\text{or} \quad WL\sin(\theta+\phi) - Mg\tfrac{L}{2}\cos\theta = 0 \qquad (4) \quad \text{about } \boxed{B}$$

The minimum θ requires the maximal frictional force

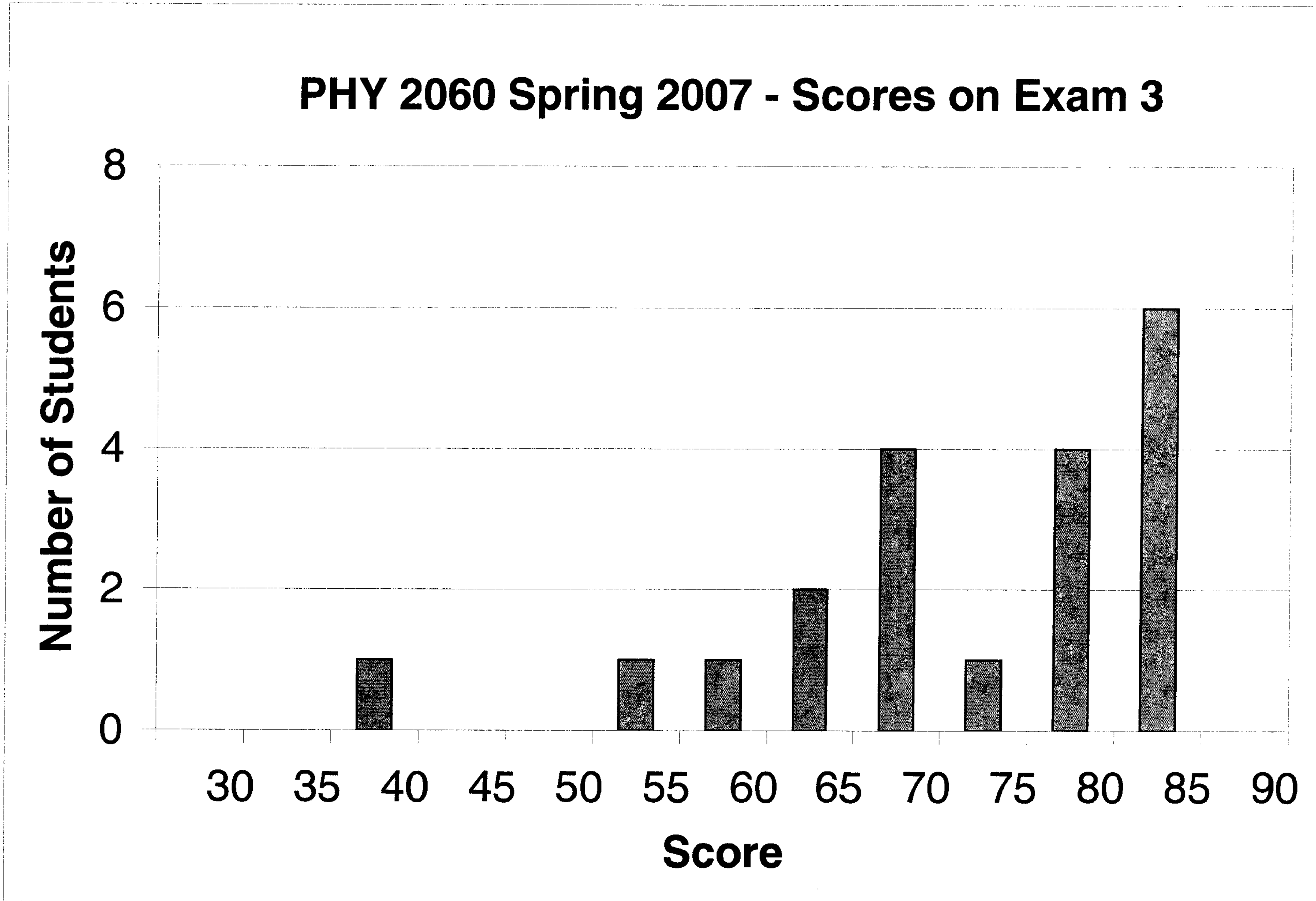
$$f = \mu_s N$$

in which case (1) $\Rightarrow$
$$W = \mu_s N \sec\phi$$

Then (2) $\Rightarrow$
$$N = \frac{Mg}{1+\mu_s\tan\phi}$$

and (3) $\Rightarrow$
$$\tan\theta = \frac{N - \frac{1}{2}Mg}{\mu_s N}$$

or
$$\theta = \text{atan}\left[\frac{1}{2}\left(\frac{1}{\mu_s} - \tan\phi\right)\right]$$


PHY 2060 Spring 2007 - Scores on Exam 3
Number of Students
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2
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Score