Kinematics of Circular Motion

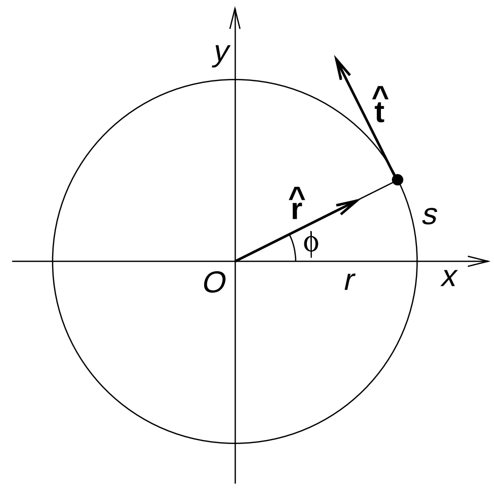
Let us analyze the kinematics of a body moving in the xy plane around the perimeter of a circle of radius r centered on the origin.

- We can describe the body's position at time t using the angle ϕ measured in a counterclockwise direction from the positive x axis. If ϕ is specified in radians, then the arc length round the perimeter of the circle in a clockwise direction from the positive x axis to the body is

$$s = r\phi,$$

and the body's speed is

$$v = \frac{ds}{dt} = r\,\frac{d\phi}{dt}.$$



- The body's position can be also described by a vector $\mathbf{r}$ having Cartesian coordinates

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Differentiating with respect to time,

$$v_x = -r\sin\phi\,\frac{d\phi}{dt} = -v\sin\phi, \qquad v_y = r\cos\phi\,\frac{d\phi}{dt} = v\cos\phi.$$

Thus, the body's velocity is

$$\mathbf{v} = v\,\hat{\mathbf{t}},$$

where the *tangential unit vector*

$$\hat{\mathbf{t}} = (-\sin\phi,\ \cos\phi)$$

is a vector of length one perpendicular to the *radial unit vector*

$$\hat{\mathbf{r}} = (\cos\phi,\ \sin\phi).$$

- Differentiating again with respect to time,

$$\begin{aligned} a_x &= -v\cos\phi\,\frac{d\phi}{dt} - \frac{dv}{dt}\,\sin\phi = -\frac{v^2}{r}\cos\phi - \frac{dv}{dt}\,\sin\phi, \\ a_y &= -v\sin\phi\,\frac{d\phi}{dt} + \frac{dv}{dt}\,\cos\phi = -\frac{v^2}{r}\sin\phi + \frac{dv}{dt}\,\cos\phi. \end{aligned}$$

Thus, the body's acceleration is

$$\mathbf{a} = -\frac{v^2}{r}\,\hat{\mathbf{r}} + \frac{dv}{dt}\,\hat{\mathbf{t}}.$$

The acceleration has a radial component $-v^2/r$ and a tangential component dv/dt.

- In the special case of *uniform circular motion*, the body's speed is constant and the acceleration $\mathbf{a} = -(v^2/r)\,\hat{\mathbf{r}}$ is purely radial.