Two-Body Collisions and the Coefficient of Restitution

Conservation of linear momentum applied to a collision between particles 1 and 2 gives

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}.$$

In d spatial dimensions, this constraint provides d independent equations, which by themselves are insufficient to fully specify the $2d$ Cartesian components of the final velocities $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ in terms of the (assumed known) quantities m_1, m_2, $\mathbf{v}_{1i}$, and $\mathbf{v}_{2i}$.

Additional information about a collision is provided by the *coefficient of restitution*,

$$\varepsilon = \frac{\text{relative speed of separation of the particles}}{\text{relative speed of approach of the particles}} = \frac{|\mathbf{v}_{2f} - \mathbf{v}_{1f}|}{|\mathbf{v}_{2i} - \mathbf{v}_{1i}|}.$$

The coefficient of restitution is a dimensionless, non-negative number that measures the degree of elasticity of the colliding bodies.

As shown in class, the coefficient of restitution between a sphere and the flat surface of a second body can be measured using free fall under gravity. The flat surface is attached rigidly in a horizontal orientation, the sphere is released from rest at a height h_i above the surface, and the maximum height h_f reached by the sphere after it strikes the surface is measured. Then $\varepsilon = \sqrt{h_f/h_i}$, a value that is independent of the acceleration due to gravity and should be the same for all values of h_i.

The coefficient of restitution can be used to classify two-body collisions into four categories:

- If $\varepsilon = 1$, the collision is *elastic*. We will see later that an elastic collision is the only type in which kinetic (or motion) energy is conserved.
- If $\varepsilon = 0$, the collision is *totally inelastic*. The particles stick together after the collision, so they have a common final velocity
$$\mathbf{v}_{1f} = \mathbf{v}_{2f} = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2}. \tag{1}$$
- If $0 < \varepsilon < 1$, the collision is *inelastic*. This is the most common case, but it is algebraically messy to handle, so it rarely appears in homework or exam questions.
- If $\varepsilon > 1$, the collision is *explosive*. An explosive collision results in an increase in the kinetic energy of the system, a situation that can arise only if some additional source of energy is present. This case is algebraically messy except for cases when an object is blown apart (e.g., the soda cans + firecracker demonstration). In such cases, $\mathbf{v}_{1i} = \mathbf{v}_{2i}$ but $\mathbf{v}_{1f} \neq \mathbf{v}_{2f}$, so $\varepsilon = \infty$.

In one dimension, the momentum-conservation equation $m_2v_{2f} + m_1v_{1f} = m_2v_{1i} + m_1v_{1i}$ can be combined with the definition of the coefficient of restitution, $v_{2f} - v_{1f} = \varepsilon(v_{1i} - v_{2i})$, to obtain

$$v_{1f} = \frac{(m_1 - \varepsilon m_2)v_{1i} + m_2(1+\varepsilon)v_{2i}}{m_1 + m_2}, \qquad v_{2f} = \frac{m_1(1+\varepsilon)v_{1i} + (m_2 - \varepsilon m_1)v_{2i}}{m_1 + m_2}. \tag{2}$$

For $\varepsilon = 0$, Eqs. (2) reduce to Eq. (1), while for $\varepsilon = 1$ they reproduce the equations for one-dimensional elastic collisions given in the text.