PHZ 3113 Fall 2011 — Exam 1

1.
$$a_n = \frac{(x-1)^n}{\sqrt{n}}$$
$$\frac{a_{n+1}}{a_n} = \sqrt{\frac{n}{n+1}}\,(x-1)$$
$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1|$$

By the ratio test, the series converges for $|x-1|<1$.

Lower end point ($x=0$): $a_n = \frac{(-1)^n}{\sqrt{n}}$ converges by Leibniz test

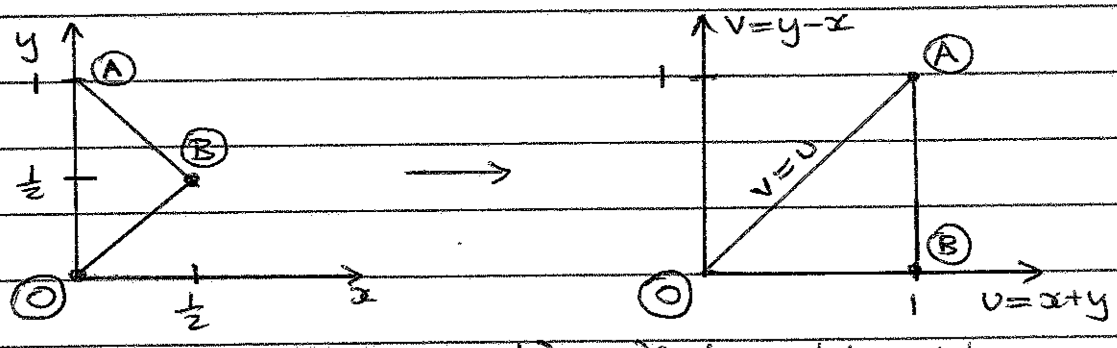
Upper endpoint ($x=2$): $a_n = \frac{1}{\sqrt{n}}$ diverges by integral test

$\Rightarrow$ Interval of convergence is $0 \le x < 2$

2(a) Implicit partial differentiation:
$$v-b + p\left(\frac{\partial v}{\partial p}\right)_T = RT\,e^{-a/RTv}\,\frac{a}{RTv^2}\left(\frac{\partial v}{\partial p}\right)_T$$
$$= p(v-b)\,\frac{a}{RTv^2}\left(\frac{\partial v}{\partial p}\right)_T$$
$$\Rightarrow \qquad p\left[1-\left(1-\frac{b}{v}\right)\frac{a}{RTv}\right]\left(\frac{\partial v}{\partial p}\right)_T = -v\left(1-\frac{b}{v}\right)$$
$$B_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T = \frac{1}{p}\,\frac{1-b/v}{1-(1-b/v)\,a/RTv}$$

(b) For $0 < \frac{b}{v}, \frac{a}{RTv} \ll 1$,
$$B_T = \frac{1}{p}\left(1-\frac{b}{v}\right)\left[1+\frac{a}{RTv}+O\left(\frac{1}{v^2}\right)\right]$$
$$= \frac{1}{p}\left[1+\left(\frac{a}{RT}-b\right)\frac{1}{v}+O\left(\frac{1}{v^2}\right)\right]$$

3.



Jacobian
$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\int_0^{1/2} dx \int_x^{1-x} dy \sqrt{\frac{y-x}{x+y}} = \int_0^1 du \int_0^u dv \sqrt{\frac{v}{u}}\left|J\left(\frac{x,y}{u,v}\right)\right| du\,dv$$
$$= \frac{1}{2}\int_0^1 du\,\frac{1}{\sqrt{u}}\left[\frac{2}{3}v^{3/2}\right]_0^u$$
$$= \frac{1}{3}\int_0^1 u\,du = \frac{1}{6}$$

4. Want to minimize $A = b^2 + 4bh$

for fixed $V = b^2 h$

$\Rightarrow$ Minimize $F = A + \lambda V = b^2 + 4bh + \lambda b^2 h$

$$0 = \frac{\partial F}{\partial b} = 2b + 4h + 2\lambda bh \quad (1)$$

$$0 = \frac{\partial F}{\partial h} = 4b + \lambda b^2 \quad (2)$$

(2) $\Rightarrow$ $\lambda = -\frac{4}{b}$ (3)

(1), (3) $\Rightarrow$ $0 = 2b + 4h - 8h$

$$\frac{h}{b} = \frac{1}{2}$$

5(a) Small change $df = \vec{\nabla} f \cdot d\vec{r} = |\vec{\nabla} f|\,|d\vec{r}| \cos\theta$

$$|df| \leq |\vec{\nabla} f|\,|d\vec{r}|$$

or $|d\vec{r}| \geqslant \frac{|df|}{|\vec{\nabla} f|}$

Here, $\vec{\nabla} f = (2z, 2y, 2x)$

$= (6, 0, 8)$ at $\vec{r} = (4, 0, 3)$

$$|\vec{\nabla} f| = \sqrt{6^2 + 8^2} = 10$$

So smallest $|d\vec{r}|$ for $|df| = 10^{-3}$ is

$$|d\vec{r}|_{min} = \frac{10^{-3}}{10} = 10^{-4}$$

(b) A plane can be defined as the set of points $\vec{r}$ satisfying

$$\vec{r} \cdot \vec{p} = c$$

For the perpendicular vector we can use

$\vec{p} = \vec{\nabla} f = (2z, 2y, 2x)$ from (a)

$= (2, 4, 6)$ at $\vec{r} = (3, 2, 1)$

To calculate c, we can use $\vec{r} = (3, 2, 1)$ since it is on the plane:

$$\vec{r} \cdot \vec{p} = 3 \times 2 + 2 \times 4 + 1 \times 6 = 20$$

$\Rightarrow$ A general point $\vec{r} = (x, y, z)$ on the plane satisfies

$$2x + 4y + 6z = 20$$

or $x + 2y + 3z = 10$