PHZ 3113 Fall 2011 – Exam 2

1. $$\int_0^2 \left[x\,\delta(x) + x^3\delta(x^2-2) - \Theta(-x) + 3x^2\Theta(x-1)\right]dx$$
$$= \int_0^2 \left[0 + x^3\frac{\delta(x-\sqrt{2})}{|2\sqrt{2}|} + x^3\frac{\delta(x+\sqrt{2})}{|-2\sqrt{2}|}\right]dx$$
$$- \int_0^0 dx + \int_1^2 3x^2\,dx$$
$$= (\sqrt{2})^3\cdot\frac{1}{2\sqrt{2}} + \left[x^3\right]_1^2$$
$$= 8$$

2. $$I = \int_{-\infty}^{+\infty}\frac{\cos\pi x\,dx}{x^2-4x+5} = \mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{i\pi x}dx}{x^2-4x+5}$$

We can convert this to a contour integral, closing the loop on a semicircle in the upper half plane where the integrand vanishes as $|z|\to\infty$.

The integrand has poles where

$$z^2-4z+5 = 0$$
$$z = 2\pm\sqrt{2^2-5} = 2\pm i$$

The simple pole at $z=2+i$ lies inside the contour and has residue

$$R(2+i) = \lim_{z\to 2+i}(z-2-i)\,f(z) = \left.\frac{e^{i\pi z}}{z-2+i}\right|_{z=2+i}$$
$$= \frac{e^{-\pi}}{2i} \qquad (\text{using } e^{2\pi i}=1)$$

By the residue theorem

$$I = \mathrm{Re}\left[2\pi i\times(\text{sum of residues enclosed})\right]$$
$$= \mathrm{Re}\left(2\pi i\,\frac{e^{-\pi}}{2i}\right)$$
$$= \pi e^{-\pi}$$

3(a) Look for $$e^{nz} = -1 = e^{i(\pi+2\pi m)} \qquad m = \text{integer}$$
$$z = (2m+1)\frac{\pi}{n}\,i$$

(b) Can write $$f(z) = \frac{g(z)}{h(z)}$$

where $$g(z) = z^2e^z$$

and $$h(z) = e^{2z}+1 \implies h'(z) = 2e^{2z} = 2\left[f(z)-1\right]$$

From (a), $h(z)$ has zeros at

$$z_m = (2m+1)\frac{\pi}{2}\,i \qquad m = \text{integer}$$

Since $\qquad g(z_m) = -(2m+1)^2 \frac{\pi^2}{4} e^{i(2m+1)\frac{\pi}{2}} = (-1)^{m+1}(2m+1)^2 \frac{\pi^2}{4} i \neq 0$

and $\qquad h'(z_m) = -2 \neq 0$

these are simple poles with residues

$$R(z_m) = \frac{g(z_m)}{h'(z_m)} = (-1)^m (2m+1)^2 \frac{\pi^2}{8} i.$$

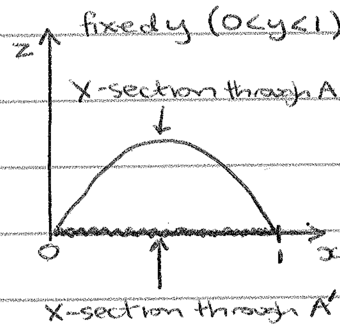
4. Gauss's theorem: $\int_{\partial V} \vec{F}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{F}\, dv$

where ∂V is the closed surface bounding volume V.

Here $\qquad \vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$

$\Rightarrow \qquad \int_{\partial V} \vec{F}\cdot d\vec{a} = 0$

Let us apply this to the volume V enclosed by the surface A and the region A' of the plane $z=0$ satisfying $0 \leq x \leq 1$, $0 \leq y \leq 1$.



$$\int_{\partial V} \vec{F}\cdot d\vec{a} = \int_A \vec{F}\cdot d\vec{a} + \int_{A'} \vec{F}\cdot d\vec{a} = 0$$

$$\Rightarrow \quad \Phi = \int_A \vec{F}\cdot d\vec{a} = -\int_{A'} \vec{F}\cdot d\vec{a} = \int_{A'} \vec{F}\cdot(-d\vec{a})$$

$$= \int_0^1 dx \int_0^1 dy\,(2x+y)$$

$$= \int_0^1 dx\,(2x+\tfrac{1}{2})$$

$$= \frac{3}{2}$$

5. Suppose $\qquad \vec{F} = \vec{\nabla}\times\vec{A}$

$$(0,0,2x+y) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

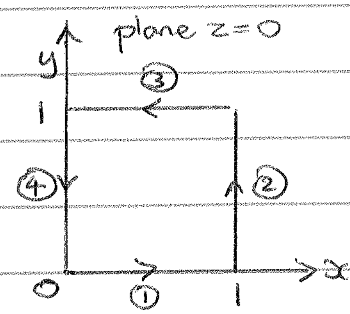
One solution (among many) is

$$\vec{A} = (-\tfrac{1}{2}y^2, x^2, 0)$$

Now we can apply Stokes' theorem

$$\int_A \vec{F}\cdot d\vec{a} = \int_A \vec{\nabla}\times\vec{A}\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{l}$$

where C is the boundary of A.



$$\Rightarrow \quad \Phi = \int_A \vec{F}\cdot d\vec{a} = \int_0^1 \overset{0}{\cancel{A_x(y=z=0)}}\,dx \quad ①$$

$$+ \int_0^1 A_y(x=1, z=0)\,dy \quad ②$$

$$+ \int_1^0 A_x(y=1, z=0)\,dx \quad ③$$

$$+ \int_1^0 \overset{0}{\cancel{A_y(x=z=0)}}\,dx \quad ④$$

$$= \int_0^1 dy - \int_0^1 (-\tfrac{1}{2})\,dx$$

$$= \frac{3}{2}$$