PHZ 3113 Fall 2011 – Exam 3

1.
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$
$$\Rightarrow \quad \sin(a\sigma_j\sigma_k) = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!}(\sigma_j\sigma_k)^{2k+1}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!}\left(I\delta_{jk} + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l\right)^{2k+1}.$$

For given j and k, <u>either</u> $\delta_{jk} \neq 0$ <u>or</u> there is one l for which $\epsilon_{jkl} \neq 0$.
$$\Rightarrow \quad \sin(a\sigma_j\sigma_k) = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!}\left[I\delta_{jk} + \sum_{l=1}^{3}(i\epsilon_{jkl}\sigma_l)^{2k+1}\right].$$

But $(i\epsilon_{jkl}\sigma_l)^{2k+1} = (-1)^k i\epsilon_{jkl}\sigma_l$, so
$$\sin(a\sigma_j\sigma_k) = \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!}\left[(-1)^k I\delta_{jk} + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l\right]$$
$$= I\sin a\,\delta_{jk} + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l \sinh a$$

2(a). The eigenvalues satisfy
$$0 = |H - \lambda I| = (3-\lambda)(1-\lambda) - (2+2i)(2-2i)$$
$$= \lambda^2 - 4\lambda - 5$$
$$\Rightarrow \quad \lambda_1 = -1, \qquad \lambda_2 = 5$$

(b) $\lambda_1 = -1$:
$$\begin{pmatrix} 4 & 2+2i \\ 2-2i & 2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i-1 \end{pmatrix}$$

$\lambda_2 = 5$:
$$\begin{pmatrix} -2 & 2+2i \\ 2-2i & -4 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

(c)
$$c_1 = \vec{v}_1^{\,\dagger}\cdot\vec{b} = \frac{1}{3\sqrt{3}}\left[1(2i) + (-i-1)(2-i)\right] = \frac{i-3}{3\sqrt{3}}$$
$$c_2 = \vec{v}_2^{\,\dagger}\cdot\vec{b} = \frac{1}{3\sqrt{6}}\left[2(2i) + (1+i)(2-i)\right] = \frac{3+5i}{3\sqrt{6}}$$

3. Since $f(x) = -f(-x)$, the cosine coefficients vanish.

Sine coefficient:
$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{2n\pi x}{2\pi}\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}\int_{\pi/2}^{\pi}\sin nx\,dx$$
$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi/2}^{\pi}$$
$$= \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$b_n = \begin{cases} \dfrac{2}{n\pi} & n \text{ odd} \\ -\dfrac{4}{n\pi} & n \text{ even}, \ \dfrac{n}{2} \text{ odd} \end{cases}$$

$$\Rightarrow \qquad f(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \left[\frac{\sin(2k+1)x}{2k+1} - \frac{\sin(4k+2)x}{2k+1} \right]$$

4. Let $$T(x,y) = T_1 + u(x,y)$$
where $$\nabla^2 u = 0.$$
Applying seperation of variables
$$u(x,y) = X(x)\,Y(y),$$
the form of $X(x)$ that satisfies the boundary conditions $X(0) = X(W) = 0$ is
$$X(x) = \sin\frac{n\pi x}{W}$$
just as in the examples considered in class and in Homework 12. The corresponding $Y(y)$ can be written
$$Y(y) = a_n \sinh\frac{n\pi y}{W} + b_n \sinh\frac{n\pi(L-y)}{W}$$
The general solution for $u(x,y)$ is therefore
$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{W} \left[a_n \sinh\frac{n\pi y}{W} + b_n \sinh\frac{n\pi(L-y)}{W} \right]$$

For $y = 0$
$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} b_n \sinh\frac{n\pi L}{W} \sin\frac{n\pi x}{W} \\ &= T_2 \sin\frac{2\pi x}{W} \end{aligned}$$
$$\Rightarrow \qquad b_n = \begin{cases} T_2 \operatorname{csch}\dfrac{2\pi L}{W} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

For $y = L$
$$\begin{aligned} u(x,L) &= \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi L}{W} \sin\frac{n\pi x}{W} \\ &= -T_3 \sin\frac{3\pi x}{W} \end{aligned}$$
$$\Rightarrow \qquad a_n = \begin{cases} -T_3 \operatorname{csch}\dfrac{3\pi L}{W} & n = 3 \\ 0 & n \neq 3 \end{cases}$$

Finally
$$\begin{aligned} T(x,y) = T_1 &+ T_2 \operatorname{csch}\frac{2\pi L}{W} \sinh\frac{2\pi(L-y)}{W} \sin\frac{2\pi x}{W} \\ &- T_3 \operatorname{csch}\frac{3\pi L}{W} \sinh\frac{3\pi y}{W} \sin\frac{3\pi x}{W} \end{aligned}$$