PHZ 3113 Fall 2012 – Exam 1

1. Distance-squared from (x,y) to $(3,-1)$ is

$$r^2 = (x-3)^2 + (y+1)^2.$$

To maximize r^2 subject to the constraint

$$x^2 + y^2 = 4 \qquad (1)$$

we can maximize

$$F = (x-3)^2 + (y+1)^2 + \lambda(x^2+y^2)$$

$$0 = \left(\frac{\partial F}{\partial x}\right)_y = 2(x-3) + 2\lambda x \qquad \Rightarrow \qquad \lambda = \frac{3}{x} - 1 \qquad (2)$$

$$0 = \left(\frac{\partial F}{\partial y}\right)_x = 2(y+1) + 2\lambda y \qquad (3)$$

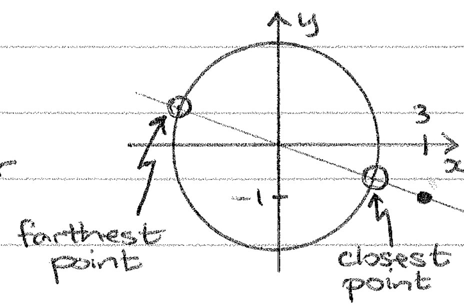
$$(2),(3) \Rightarrow \quad 0 = y + 1 + \left(\frac{3}{x} - 1\right) y \qquad \Rightarrow \qquad y = -\frac{x}{3} \qquad (4)$$

$$(1),(4) \Rightarrow \quad 4 = x^2 + \left(-\frac{x}{3}\right)^2 = \frac{10}{9} x^2$$

$$x = \pm \frac{6}{\sqrt{10}}, \qquad y = -\frac{x}{3}$$

Clearly, the solution $x < 0$ gives the maximal r

$$(x,y) = \frac{1}{\sqrt{10}}(-6, 2)$$



2(i) If $\quad S(x) = \sum_{n=1}^{\infty} a_n$

then $\quad \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \left(\frac{n+1+r}{n+r}\right)^2 \frac{x}{\pi} = \frac{x}{\pi}$

Ratio test $\Rightarrow$ $S(x)$ converges for $|\rho| < 1$, diverges for $|\rho| > 1$.

(ii) For $x = -\pi$, $\quad a_n = (-1)^n \frac{n}{(n+r)^2} \xrightarrow{n\to\infty} \frac{(-1)^n}{n}$ so $|a_{n+1}| < |a_n|$ and $a_n \to 0$.

Leibniz test $\Rightarrow$ $S(-\pi)$ converges.

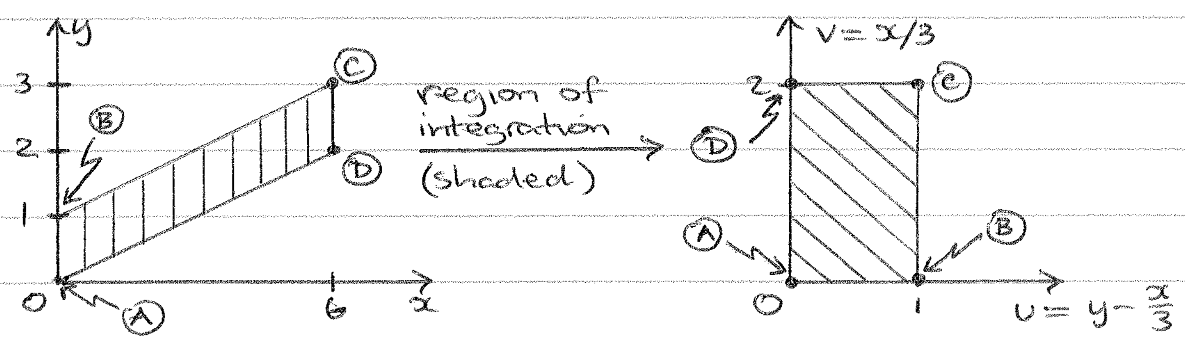
(iii) For $x = \pi$, $\quad \frac{a_{n+1}}{a_n} = \frac{n+1}{n}\left(\frac{n+r}{n+1+r}\right)^2 = \left(1+\frac{1}{n}\right)\left(\frac{1+r/n}{1+\frac{1+r}{n}}\right)^2$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{2r}{n}+\frac{r^2}{n^2}\right)\left(1-2\,\frac{1+r}{n}+O\!\left(\frac{1}{n^2}\right)\right) = 1 - \frac{1}{n} + O\!\left(\frac{1}{n^2}\right)$$

Gauss test $\Rightarrow$ $S(\pi)$ diverges. (Or use comparison test with $\sum_n \frac{c}{n}$, $c > 1$.)

(iv) Summary: Interval of convergence is $-\pi \le x < \pi$ for all $r > 0$.

3.



Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -1/3 & 1 \\ 1/3 & 0 \end{vmatrix} = -\frac{1}{3} = 1/J\left(\frac{x,y}{u,v}\right)$$

$$\Rightarrow \int_0^6 dx \int_{x/3}^{x/3+1} dy \; x\left(y-\frac{x}{3}\right)^2$$

$$= \int_0^1 du \int_0^2 dv \left|J\left(\frac{x,y}{u,v}\right)\right| 3v\,u^2$$

$$= 9\int_0^1 u^2\,du \int_0^2 v\,dv = 9\left[\tfrac{1}{3}u^3\right]_0^1 \left[\tfrac{1}{2}v^2\right]_0^2$$

$$= 6$$

4(a) $$V = \frac{RT}{p}$$

$$\alpha_{ideal} = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{V}\frac{R}{p} = \frac{1}{T}$$

(b) $$V = \frac{RT}{p}\left(1-\frac{V_c}{V}\right)$$

Either $$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{p}\left(1-\frac{V_c}{V}\right) + \frac{RT}{p}\frac{V_c}{V^2}\left(\frac{\partial V}{\partial T}\right)_p = \frac{V}{T} + \left(1-\frac{V_c}{V}\right)^{-1}\frac{V_c}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

$$\Rightarrow \alpha_{non\text{-}ideal} = \frac{1}{T} + \frac{V_c}{V-V_c}\alpha_{non\text{-}ideal} = \frac{V-V_c}{V-2V_c}\frac{1}{T}$$

or $$T = \frac{pV}{R(1-V_c/V)}$$

$$\left(\frac{\partial T}{\partial V}\right)_p = \frac{p}{R(1-V_c/V)} - \frac{pV}{R(1-V_c/V)^2}\frac{V_c}{V^2} = \frac{T}{V} - \frac{T}{V}\frac{V_c}{V-V_c} = \frac{T}{V}\frac{V-2V_c}{V-V_c}$$

$$\Rightarrow \alpha_{non\text{-}ideal} = \left[V\left(\frac{\partial T}{\partial V}\right)_p\right]^{-1} = \frac{V-V_c}{V-2V_c}\frac{1}{T}$$

(c) For $V > 2V_c$, $\frac{V-V_c}{V-2V_c} > 1 \quad \Rightarrow \quad \alpha_{non\text{-}ideal} > \alpha_{ideal}$

5(a) $$v = x^2y + xy^2 \Rightarrow \quad 0 = 2xy + y^2 + (x^2+2xy)\left(\frac{\partial y}{\partial x}\right)_v$$ implicit differentiation

$$\left(\frac{\partial y}{\partial x}\right)_v = -\frac{2xy+y^2}{x^2+2xy}$$

(b)

$$u = x+y \Rightarrow \left(\frac{\partial u}{\partial v}\right)_w = \left(\frac{\partial x}{\partial v}\right)_w + \left(\frac{\partial y}{\partial v}\right)_w \qquad (1)$$

$$v = x^2y+xy^2 \Rightarrow 1 = (2xy+y^2)\left(\frac{\partial x}{\partial v}\right)_w + (x^2+2xy)\left(\frac{\partial y}{\partial v}\right)_w \qquad (2)$$

$$w = x^3-y^3 \Rightarrow 0 = 3x^2\left(\frac{\partial x}{\partial v}\right)_w - 3y^2\left(\frac{\partial y}{\partial v}\right)_w \qquad (3)$$

$$(3) \Rightarrow \left(\frac{\partial y}{\partial v}\right)_w = \frac{x^2}{y^2}\left(\frac{\partial x}{\partial v}\right)_w \qquad (4)$$

$$(2),(4) \Rightarrow 1 = \left[2xy+y^2+(x^2+2xy)\frac{x^2}{y^2}\right]\left(\frac{\partial x}{\partial v}\right)_w \qquad (5)$$

$$(1),(4),(5) \Rightarrow \left(\frac{\partial u}{\partial v}\right)_w = \frac{x^2+y^2}{(x+2y)x^3+(y+2x)y^3}$$