PHZ 3113 Fall 2012 – Exam 2

[15] 1. Gauss's theorem: $\int_{\partial V} \vec{F}\cdot d\vec{a} = \int_V \vec{\nabla}\vec{F}\,dv$ where ∂V is the closed surface bounding volume V

Here, $\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 3x^2 + 2y$

$$\Rightarrow \quad \int_{\partial V} \vec{F}\cdot d\vec{a} = \int_{-1}^{1} dx \int_0^2 dy \int_1^3 dz\,(3x^2+2y)$$
$$= 3\int_{-1}^{1} x^2 dx \int_0^2 dy \int_1^3 dz + 2\int_{-1}^{1} dx \int_0^2 y\,dy \int_1^3 dz$$
$$= 3\cdot\frac{2}{3}\cdot 2\cdot 2 + 2\cdot 2\cdot 2\cdot 2$$
$$= 24$$

[10] 2(a) $$\int_0^5 \left[e^x\delta(x^3-8) + x\,\Theta(3-x)\right]dx = \int_0^5 e^x\,\frac{\delta(x-2)}{3x^2}\,dx + \int_0^3 x\,dx$$
$$= \frac{e^2}{12} + \frac{9}{2}$$

[10] (b) $$\int \cos^2\theta\,\delta(r-a)\,dv = \int_0^\infty r^2 dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\,\cos^2\theta\,\delta(r-a)$$
$$= \int_0^\infty r^2\delta(r-a)\,dr \int_{-1}^{1} \mu^2 d\mu \int_0^{2\pi} d\phi \quad (\mu \equiv \cos\theta)$$
$$= a^2\,\Theta(a)\cdot\frac{2}{3}\cdot 2\pi$$
$$= \frac{4\pi}{3}a^2\,\Theta(a).$$

[10] 3(a) $$z_1 = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3}+i$$
$$z_1 + z_2 = \sqrt{3}+i+1+\sqrt{3}\,i = (\sqrt{3}+1)(1+i)$$
$$= \sqrt{2}(\sqrt{3}+1)\,e^{i(\pi/4 + 2\pi m)}, \quad m = \text{any integer}$$
$$(z_1+z_2)^{1/3} = 2^{1/6}(\sqrt{3}+1)^{1/3}\,e^{i\left(\frac{\pi}{12} + \frac{2\pi}{3}m\right)} \quad \text{(3 distinct solutions)}$$

[10] (b) $$I = \oint_c \frac{\sinh z}{3\pi + 2iz}\,dz = \frac{1}{2i}\oint \frac{\sinh z}{z - \frac{3\pi i}{2}}\,dz$$

Since $z = \frac{3\pi i}{2}$ lies inside the contour, by Cauchy's integral formula,

$$I = 2\pi i\,\frac{1}{2i}\sinh\frac{3\pi i}{2}$$
$$= \pi\,\frac{1}{2}\left(e^{3\pi i/2} - e^{-3\pi i/2}\right)$$
$$= i\pi\sin\frac{3\pi}{2}$$
$$= -\pi i$$

[10] 4(a) Let

$$z = 2+w$$

$$\Rightarrow \quad f(z) = \frac{1}{w^2(2+w)} = \frac{1}{2w^2(1+w/2)}$$

$$= \frac{1}{2w^2}\sum_{k=0}^{\infty}\left(-\frac{w}{2}\right)^k \qquad \text{for } |w|<2.$$

$$= \tfrac{1}{2}(z-2)^{-2} - \tfrac{1}{4}(z-2)^{-1} + \tfrac{1}{8} - \tfrac{1}{16}(z-2) + \dots$$

$$\Rightarrow \quad a_{-2} = \tfrac{1}{2}, \quad a_{-1} = -\tfrac{1}{4}, \quad a_0 = \tfrac{1}{8}, \quad a_1 = -\tfrac{1}{16}.$$

[10] (b) $f(z)$ has a pole of order 2 at $z=2$. From (a), the residue is

$$R(2) = a_{-1} = -\tfrac{1}{4}.$$

$f(z)$ also has a simple pole (ie, order 1) at $z=0$. The residue is

$$R(0) = \lim_{z\to 0} z\, f(z) = \lim_{z\to 0}\frac{1}{(z-2)^2}$$

$$= \tfrac{1}{4}$$

[25] 5. We want to find a vector field $\vec{V}$ such that

$$\vec{\nabla}\times\vec{V} = \vec{F}$$

$$\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z},\ \frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x},\ \frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right) = (0, 0, x^2+y^2).$$

One such field is

$$\vec{V} = \left(-\tfrac{1}{3}y^3, \tfrac{1}{3}x^3, 0\right)$$

Now we can apply Stokes' theorem:

$$\Phi = \int_A \vec{F}\cdot d\vec{a} = \int_A \vec{\nabla}\times\vec{V}\cdot d\vec{a} = \int_{\partial A}\vec{V}\cdot d\vec{r}$$

where ∂A is the closed curve bounding A.

$$\Phi = \int_0^2 \cancelto{0}{V_x(y=0, z=0)}\,dx \quad ①$$

$$+ \int_0^1 V_y(x=2, z=0)\,dy \quad ②$$

$$+ \int_2^0 V_x(y=1, z=0)\,dx \quad ③$$

$$+ \int_1^0 \cancelto{0}{V_y(x=0, z=0)}\,dx \quad ④$$

$$= \int_0^1 \tfrac{8}{3}\,dy - \int_0^2\left(-\tfrac{1}{3}\right)dx$$

$$= \frac{10}{3}$$