PHZ 3113 Fall 2012 – Homework 5

Due at the start of class on Friday, October 5. Half credit will be available for homework submitted after the deadline but no later than the start of class on October 8.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

Throughout the questions below, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along the Cartesian x, y, and z axes, respectively.

1. Consider the scalar field $\phi(\mathbf{r}) = 1/|\mathbf{r}|^2 - (\hat{\mathbf{x}}\cdot\mathbf{r})^2/|\mathbf{r}|^4$. Your answers to (a) and (b) below may contain the Cartesian components x, y, z of $\mathbf{r}$ and/or $r = |\mathbf{r}|$.

   (a) Find the Cartesian components of $\boldsymbol{\nabla}\phi$.

   (b) Find $\nabla^2\phi$.

   (c) Evaluate $\hat{\mathbf{y}}\cdot\boldsymbol{\nabla}\phi$, $|\boldsymbol{\nabla}\phi|$, and $\nabla^2\phi$ at $\mathbf{r} = (-2, 1, 1)$.

2. Use properties of the Levi-Civita symbol ϵ_{ijk} to show that

   (a) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$, (b) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}$.

3. Let Φ be the outward flux of the vector field $\mathbf{F} = (y-z)\,\hat{\mathbf{x}} + x^2y\,\hat{\mathbf{y}}$ through the region $z > 0$ of the surface of a sphere of radius R centered on the origin.

   (a) Calculate Φ by direct evaluation of the flux.

   (b) Calculate Φ by converting the open surface to a closed one and applying the divergence theorem.

4. Let Γ be the circulation of the vector field with Cartesian coordinates $(x^2yz^3, 0, 3x^3y^2z)$ around the perimeter of the ellipse $9x^2+y^2 = 1$, $z = 1$, traveling counterclockwise when viewed from positive values of z.

   (a) Calculate Γ by direct evaluation of the circulation.

   (b) Calculate Γ using Stokes' theorem.

   Hint: Change variables using $x = \frac{1}{3}r\cos\phi$ and $y = r\sin\phi$.

5. Green's theorem states that if p and q are functions of x and y having continuous first partial derivatives within some region A of the x-y plane with boundary C, then

$$\oint_C (pdx + qdy) = \int_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy, \tag{1}$$

   where the left-hand integral is taken counterclockwise around the entire boundary.

Equation (1) can be regarded as a special case of Stokes' theorem for a vector field with Cartesian components $(p, q, 0)$ where A is confined to lie in a plane of constant z. It can also be derived from the two-dimensional version of the divergence theorem,

$$\int_A \boldsymbol{\nabla} \cdot \mathbf{F} \, da = \oint_C \mathbf{F} \cdot (d\mathbf{r} \times \hat{\mathbf{z}})$$

(where $d\mathbf{r}$ is an infinitesimal vector along the boundary C and $d\mathbf{r} \times \hat{\mathbf{z}}$ is therefore a vector of length $|d\mathbf{r}|$ pointing along the outward normal direction) by setting $\mathbf{F} = (q, -p)$.

As one application of Green's theorem, the area of a region A can be found by evaluating the line integral $\oint_C (p dx + q dy)$ for any p and q such that $\partial q/\partial x - \partial p/\partial y = 1$, e.g.,

$$\text{area} = \oint_C x dy = -\oint_C y dx = \frac{1}{2} \oint_C (x dy - y dx).$$

(a) Use Green's theorem to evaluate the line integral $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for $\mathbf{v} = xy \, \hat{\mathbf{x}} - 3x^2 \hat{\mathbf{y}}$ and a path C taken counterclockwise around the edge of a triangle with vertices at $(x, y) = (0, -1)$, $(2, 0)$, and $(0, 2)$.

(b) Use Green's theorem to find the area of the ellipse parametrized by $x = a \cos\phi$, $y = b \sin\phi$ for $0 \leq \phi < 2\pi$.

(c) Use Green's theorem to find the area of the shape enclosed by the parametric equations $x = a \cos\phi$, $y = b \sin^3\phi$ for $0 \leq \phi < 2\pi$. Sketch the shape for the case $a = 2b$.