PHZ 3113 Fall 2012 – Homework 6

Due at the start of class on Friday, October 12. Half credit will be available for homework submitted after the deadline but no later than the start of class on Monday, October 15.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

Throughout the questions below, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along the Cartesian x, y, and z axes, respectively.

1. Evaluate

$$\int_0^\pi \sin(x)\,\delta(\cos 3x)\,dx.$$

2. There are many situations where it is useful to be able to integrate and differentiate the Dirac delta function. Based on the basic properties of $\delta(x)$, $\int_a^b \delta(x)\,dx = \Theta(b) - \Theta(a)$ where $\Theta(x)$ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0. \end{cases}$$

Loosely speaking, $\int \delta(x-c)\,dx = \Theta(x-c)$. Conversely, $d\Theta(x-c)/dx = \delta(x-c)$ and $d\Theta(c-x)/dx = -\delta(x-c)$.

   (a) Express $\int_a^b f(x)\,\delta(x)\,dx$ using the Heaviside function.

   (b) Suppose that $g(x) = x^2\Theta(2-x) - 3x\Theta(x+1)$. Sketch $g(x)$ over the range $-4 \leq x \leq 4$ and express dg/dx in terms of delta functions.

The derivative of the delta function can be treated formally as the limit $\varepsilon \to 0$ of the derivative (with respect to x) of one of the peaked functions $\delta_\varepsilon(x)$ considered in class. However, we can understand the derivative $\delta'(x) = d\delta(x)/dx$ non-rigorously by treating $\delta(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x)$ as if it were a function.

   (c) Simplify $\int_{-\infty}^\infty f(x)\,\delta'(x)\,dx$ using the definition

$$\delta'(x) = \lim_{\Delta x \to 0} \frac{\delta(x + \Delta x) - \delta(x)}{\Delta x}.$$

   (d) Simplify $\int_{-\infty}^\infty f(x)\,\delta'(x)\,dx$ assuming that you can apply integration by parts.

   (e) Simplify $\int_{-\infty}^\infty f(x)\,\delta'(a-x)\,dx$ using the method from (c) or the method from (d).

3. One family of functions that can be used to approximate the Dirac delta function is the set of Lorentzians

$$\delta_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2 + \varepsilon^2}.$$

(a) Find the corresponding approximation $\Theta_\varepsilon(x)$ to the Heaviside function.

(b) Find the corresponding approximation $\delta'_\varepsilon(x)$ to the derivative of the delta function.

(c) Sketch the qualitative forms of $\delta_\varepsilon(x)$, $\Theta_\varepsilon(x)$, and $\delta'_\varepsilon(x)$ on a graph covering the range $-5\varepsilon \le x \le 5\varepsilon$.

4. This question concerns the representation of the magnetic field in terms of a vector potential: $\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t)$. Note that both $\mathbf{B}(\mathbf{r}, t)$ and the electric field $\mathbf{E}(\mathbf{r}, t) = -\boldsymbol{\nabla}\phi(\mathbf{r}, t) - \partial\mathbf{A}(\mathbf{r}, t)/\partial t$ are unchanged by the gauge transformation $\mathbf{A}(\mathbf{r}, t) \to \mathbf{A}(\mathbf{r}, t) + \boldsymbol{\nabla}\psi(\mathbf{r}, t)$, $\phi(\mathbf{r}, t) \to \phi(\mathbf{r}, t) - \partial\psi(\mathbf{r}, t)/\partial t$, where $\psi(\mathbf{r}, t)$ is any differentiable function. The resulting ambiguity in the scalar and vector potentials can be removed by adopting a gauge-fixing condition. For example, the Coulomb gauge is defined by the condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

(a) Show that $\mathbf{A} = \frac{1}{2}\mathbf{B}_0 \times \mathbf{r}$, where $\mathbf{B}_0$ is a constant vector, satisfies the Coulomb gauge condition, and describes a spatially uniform, time-invariant magnetic field $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0$.

(b) Provide the Cartesian components of the vector potential $\mathbf{A}$ defined in (a) for the specific case $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$.

(c) Verify that the vector potentials $\mathbf{A}_1 = -B_0 y\hat{\mathbf{x}}$ and $\mathbf{A}_2 = B_0 x\hat{\mathbf{y}}$ describe the same magnetic field as $\mathbf{A}$ defined in (b). Find functions $\psi_1(\mathbf{r}, t)$ and $\psi_2(\mathbf{r}, t)$ that transform $\mathbf{A}$ to $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively.

(d) Show that $\mathbf{A} = \frac{1}{2}(f\boldsymbol{\nabla}g - g\boldsymbol{\nabla}f)$ does not in general satisfy the Coulomb gauge condition, but it does produce a magnetic field that satisfies the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Provide an expression for this field using abstract (coordinate-independent) vector notation.