PHZ 3113 Fall 2012 – Homework 9

Due at the start of class on Wednesday, November 14. Half credit will be available for homework submitted after the deadline but no later than the start of class on Friday, November 16.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Use complex integration methods to evaluate the following integrals:

$$(a)\ \int_0^\pi \frac{d\theta}{(2+\cos\theta)^2}\,. \qquad (b)\ P\int_{-\infty}^{\infty} \frac{dx}{x^2+5x+4}\,. \qquad (c)\ \int_0^\infty \frac{\sin^2 x}{x^2+1}\,dx.$$

2. Find the determinants of the following matrices. If the matrix is nonsingular, find its inverse using the formula $A^{-1} = (\det A)^{-1} C^T$ where C is the matrix of cofactors.

$$\text{(a)}\ \begin{pmatrix} 8 & -2 \\ 3 & 1 \end{pmatrix} \quad \text{(b)}\ \begin{pmatrix} 4 & -2 & -2 \\ 0 & -2 & 4 \\ -2 & 1 & 1 \end{pmatrix} \quad \text{(c)}\ \begin{pmatrix} -1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$$

3. In the quantum mechanical description of spin-$\frac{1}{2}$ particles (such as electrons, protons, and neutrons), a central role is played by the *Pauli matrices*:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this question, $i = \sqrt{-1}$; j, k, and l are indices that can range from 1 to 3, and μ is an index that ranges from 0 to 3.

(a) Verify (by brute force) that $\sigma_j \sigma_k = I\delta_{jk} + i\sum_l \epsilon_{jkl}\sigma_l$, where ϵ_{jkl} is the Levi-Cevita symbol and

$$I \equiv \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Use the result that you verified in (a) to show that the commutators of Pauli matrices satisfy $[\sigma_j,\, \sigma_k] = 2i\sum_l \epsilon_{jkl}\sigma_l$.

(c) Show that any Hermitian (self-adjoint) 2×2 matrix M can be written $M = \sum_\mu a_\mu \sigma_\mu$ where each a_μ is real.

(d) For any function $f(x)$ that has a Taylor expansion about $x = 0$, one can define the corresponding function of a square matrix M to be

$$f(M) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, M^n,$$

where $M^0 = I$. Use the result that you verified in (a) to show that

i. $\exp(\alpha\sigma_\mu)$ is convergent for any finite α (real or complex),

ii. $\ln(I + \alpha\sigma_\mu)$ is convergent for any $|\alpha| < 1$.

In each case, express the function in closed form (without any infinite sum remaining in the final answer).