PHY 4523 Spring 2001

Extra-Credit Assignment

Due by 5:00 p.m. on Friday, April 27. No credit will be offered for assignments turned in after this time. (The general amnesty for the first late homework of the semester does not apply to this assignment.)

Each question is worth 3% of the total course score. To receive full credit, you should explain your reasoning and show all working. Please write neatly and remember to include your name.

1. *More on the Debye model.*

   Do Reif Problems 10.2 and 10.3.

2. *Vibrations of a 1D diatomic chain.*

   Repeat the analysis of the vibrational modes of a 1D diatomic chain (see the handout distributed in class for a description of the model and the definition of all symbols), specializing to the case $M_1 = M_2 = M$ but $K' \neq K$.

   (a) Write down the dispersion relations $\omega_{\pm}(q)$ for the two branches of normal-mode solutions.

   (b) Expand the solutions for $\omega_-(q)$ and $\omega_+(q)$ in the limit $|qa| \ll 1$. For each branch, expand as far as the first term containing $|qa|$. Find the ratio of vibrational amplitudes A/B for each branch in the limit $q \to 0$.

   (c) Expand the solutions for $\omega_-(q)$ and $\omega_+(q)$ in the limit $|\pi - qa| \ll 1$, assuming that $K' > K$. For each branch, expand as far as the first term containing $|\pi - qa|$. Find the ratio of vibrational amplitudes A/B for each branch in the limit $q \to \pi/a$.

   (d) Repeat part (c) assuming instead that $K > K'$. Show that the ratio A/B of the upper [lower] branch corresponds to the ratio A/B of the lower [upper] branch in part (c). Provide a simple *physical* explanation why this interchange takes place.

   (e) Sketch both branches of the $\omega(q)$ curve over the range $0 < q < \pi/a$, assuming that $K' > K$. Comment on how the curves would change if the difference $K' - K$ were taken to zero (keeping $M_1 = M_2 = M$).

3. *Capillary waves on a liquid surface.* For short wavelengths, the frequency of waves on the surface of a liquid is determined primarily by the surface tension. (For longer wavelengths, gravity dominates.) Capillary waves may be treated as a set of harmonic oscillators with only one allowed polarization (with liquid moving perpendicular to the surface). The relationship between the wavevector $\mathbf{k}$ and the angular frequency ω is $\omega^2 = \sigma_0 |\mathbf{k}|^3/\rho$, where ρ is the mass density and σ_0 is the surface tension in the absence of capillary waves.

   (a) Show that the number of oscillator modes having a wavevector whose magnitude lies between k and $k + dk$ is $(A/2\pi)k dk$, where A is the area of the liquid surface.

(b) Find the density of states $D(\omega)$, such that $D(\omega)d\omega$ is the number of oscillator modes with an angular frequency between ω and $\omega + d\omega$.

(c) Neglecting the zero-point energy, derive the canonical partition function for a single quantum harmonic oscillator whose frequency is ω.

(d) Evaluate the contribution of capillary waves to $F(T, A)$, the Helmholtz free energy of the surface.

(e) Determine the contribution of capillary waves to the surface tension, $\sigma = (\partial F/\partial A)_T$.

You may find the following integral useful:

$$\int_0^\infty x^{1/3} \ln(1 - e^{-x})\, dx \approx -1.264.$$