PHY 4523 Spring 2001 – Homework 5

Due by 5:00 p.m. on Friday, March 30.

Answer both questions. To receive full credit, you should explain your reasoning and show all working. Please write neatly and remember to include your name.

1. *Quantum statistics for "doublons."* Suppose that there is a new type of particle, the "doublon," having the property that no more than two doublons may occupy the same single-particle state. In other words, the occupancy of each single particle state r can take the values $n_r = 0$, 1, or 2, with a corresponding energy $n_r \epsilon_r$.

   (a) Derive an expression for $\bar{n}_D(\epsilon)$, the expected occupancy of single-particle level of energy ϵ when the system is in equilibrium with a reservoir held at temperature T and chemical potential μ.

   (b) Let $\bar{n}_B(\epsilon)$ and $\bar{n}_F(\epsilon)$ be the expected occupancies for fermionic and bosonic particles, respectively. Assume that μ and T take the same values as in (a). Prove that $\bar{n}_B(\epsilon) \geq \bar{n}_D(\epsilon) \geq \bar{n}_F(\epsilon)$.

   (c) Compare $\bar{n}_D(\epsilon)$ with the Maxwell-Boltzmann occupancy $\bar{n}_M(\epsilon)$. Find the energy range (if any) over which the doublon occupancy is higher, and the range (if any) over which it is lower.

2. *Quantum statistics for small fermionic systems.*

   The Fermi-Dirac expression for the average occupancy of a single-particle state r in a system of N fermions is

$$\bar{n}_r = \frac{1}{e^{\beta(\epsilon_r - \mu)} + 1}, \tag{1}$$

   where μ is determined by the condition

$$\sum_r \bar{n}_r = N. \tag{2}$$

   This question asks you to test the validity of this distribution function for a system which contains exactly two fermions, each of which can be in one of three single-particle states. State 1 has energy $\epsilon_1 = -\varepsilon$, state 2 has energy $\epsilon_2 = -\varepsilon$, and state 3 has energy $\epsilon_3 = +\varepsilon$, where $\varepsilon > 0$.

   (a) Create a table listing every possible state of the two-particle system. The table should have columns for n_1, n_2, n_3 and the total energy E.

   (b) Calculate the average occupancies $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$ exactly within the canonical ensemble.

   (c) Deduce the value of the chemical potential μ implied by Eqs. (1) and (2). You should be able to write a closed-form answer $\mu = \ldots$.

   Hint: Derive a quadratic equation for $e^{-\beta\mu}$. Only one solution of this equation is physical, i.e., ensures that all $\bar{n}_r$ lie within the allowed range.

(d) Does substituting your answer to (c) into Eq. (1) reproduce the exact results from (b)? Is the Fermi-Dirac distribution valid for this system?

Hint: Comparison of the two methods for computing $\bar{n}_r$ is messy for general values of β and ε. It will be sufficient for the purposes of this problem to expand your two expressions for each $\bar{n}_r$ in the limit $e^{-\beta\varepsilon} \ll 1$, and compare the two expansions term by term.