PHY 4523 Spring 2001 – Homework 7

Due by 5:00 p.m. on Friday, April 20.

Answer both questions. To receive full credit, you should explain your reasoning and show all working. Please write neatly and remember to include your name.

1. *Entropy — Electrons vs Photons.*

   Reif Problem 9.19.

   Hint: Equation (4.4.9) may be of use in answering this question.

2. *White Dwarf Stars.*

   White dwarfs are old stars which have contracted as their nuclear fuel runs out, i.e., as their hydrogen is converted to helium. A typical white dwarf has roughly the mass of the sun and a radius similar to the earth's. The aim of this problem is to find the mass-radius relation for white dwarfs, which is determined by a balance between gravitational collapse and the pressure exerted by degenerate electrons. It turns out that the electrons must be treated relativistically, and that even though the temperature inside the star is of order 10^7 K, the electrons are in the degenerate limit $T \ll T_F$.

   Assumptions: Consider a spherical white dwarf of mass M and radius R, composed of fully ionized ^{4}He (each ^{4}He atom having dissociated into a nucleus of mass $m_n \approx 6.7 \times 10^{-27}$ kg plus two electrons, each of mass $m_e \approx 9.1 \times 10^{-31}$ kg). Take the mass of the star to be entirely due to the nuclei (neglecting the electron mass) and the pressure to arise solely from the electrons (disregarding contributions from the nuclei and from electromagnetic radiation). Furthermore, assume that both nucleons and electrons are distributed uniformly inside the star.

   (a) Calculate the electrons' Fermi momentum, $p_F = \hbar k_F$. Express your answer in terms of M, R, m_n, and standard constants.

   Hint: The relativistic nature of the electrons is not important here.

   (b) Calculate U_{el}, the total energy of the electrons. The electrons are highly degenerate, so you can take the effective temperature to be zero (i.e., $10^7\,\text{K} \approx$ absolute zero!). Since the electrons obey the relativistic energy-momentum relation $\epsilon = \sqrt{m_e^2 c^4 + p^2 c^2}$, you should not assume a density of states $\rho(\epsilon) \propto \sqrt{\epsilon}$; instead, you should calculate $U_{\text{el}} = \int dp\, \rho_p \ldots$. When performing this integration over p, it may prove convenient to make a change of variables to θ defined by

   $$p = m_e c \sinh\theta \quad \Rightarrow \quad \epsilon = m_e c^2 \cosh\theta.$$

   Your final result should be a closed-form expression in terms of M, R, m_n, m_e, and standard constants.

(c) The gravitational energy of a uniform sphere of mass M and radius R is

$$U_{\text{grav}} = -\frac{3GM^2}{5R},$$

where G is the gravitational constant. Minimize the total energy $U_{\text{el}} + U_{\text{grav}}$ to find an equation connecting R and M. The equation should contain only M, R, m_n, m_e, plus standard constants.

(d) Convert your answer to (c) to a closed-form expression for R as a function of M, valid in the limit $p_F \gg m_e c$. Show that your equation implies that there is an upper limit on the mass of a white dwarf. Evaluate the limiting mass, M_0, and compare it to the mass of the sun, $M_s \approx 2.0 \times 10^{30}$ kg.

It turns out that if $M > M_0$, the electron pressure is insufficient to prevent the star contracting to form a neutron star (a few kilometers in radius) or even a black hole.

In solving this problem, you may find the following integral useful:

$$\int \sinh^2 x \cosh^2 x \, dx = \frac{1}{32}\sinh(4x) - \frac{x}{8}.$$