PHY 4523 Spring 2001 – Sample Mid-Term Exam, Part 1

You have 50 minutes in which to attempt both problems. The maximum points for each problem are shown. **To gain full credit you must explain your reasoning and show all working.** *You are allowed to consult Reif, Callen, and/or your lecture notes but no other materials.*

1. (*10 points.*) Calculate the constant-volume heat capacity of a system of N independent, distinguishable particles held at a temperature T sufficiently high that classical statistical mechanics can be applied. Each particle has a mass m and is free to perform one-dimensional oscillations about its equilibrium position. The particle's potential energy is αx^{2n}, where x is the particle's displacement from its equilibrium position, α is a constant, and n is a positive integer.

   Hint: This situation involves a generalization of the equipartition theorem. Write down an expression for the logarithm of the partition function. Transform all integrals to dimensionless variables. Do not attempt to evaluate the integrals—they only contribute additive constants to $\ln Z$.

2. (*15 points.*) A simple "zipper" model for describing the unwinding of deoxyribonucleic acid treats each DNA molecule as a sequence of N base pairs. Each pair can be either linked (energy 0) or unlinked (energy ε). A pair can unlink only if it is at the end of the molecule or if an neighboring pair is already unlinked (just like a zipper).

   In class, we saw that if one end of the molecule is "capped" so that the molecule can unlink only from the other end, its partition function is

$$Z_{\text{1-end}} = \sum_{n=0}^{N} e^{-\beta n \varepsilon} = \frac{1 - e^{-\beta(N+1)\varepsilon}}{1 - e^{-\beta\varepsilon}}. \tag{1}$$

   In the sum, n labels the number of unlinked base pairs, which are all grouped at the uncapped end of the molecule—see Fig. 1.

   (a) Calculate $Z_{\text{2-ends}}$, the partition function for situations in which the molecule can unwind from both ends. You should perform all summations, i.e., don't leave any $\sum$ signs in your answer.

   Hint: Any allowed configuration of the molecule can be labeled by a pair of integers (n_1, n_2), where n_j is the number of unlinked based pairs at end j of the molecule—see Fig. 2. What are the allowed values of n_1 and n_2?

   (b) What should be the relation between $Z_{\text{1-end}}$ and $Z_{\text{2-ends}}$ in the limit $N \to \infty$? Explain your reasoning. Do Eq. (1) and your expression for $Z_{\text{2-ends}}$ satisfy the expected relation?

PHY 4523 Spring 2001 – Sample Mid-Term Exam, Part 2

Note: The following is the second part of the mid-term set in Spring 2000. Prior to this exam, a similar problem was distributed as part of the sample exam. The sample problem made up Homework 3 this semester.

You have 50 minutes in which to attempt this problem. **To gain full credit you must explain your reasoning and show all working.** *You are allowed to consult Reif, Callen, and/or your lecture notes but no other materials.*

A magnetic moment of magnitude μ sits at each vertex of a two-dimensional square lattice. The moment is constrained to lie in the plane of the lattice and to point directly towards one of the four nearest-neighbor vertices. There is no interaction between moments at different sites.

In zero magnetic field, the four possible orientations of each moment would have the same energy. However, consider the situation in which a uniform magnetic field of magnitude H is applied in the plane of the lattice at an angle of 45° to one of the four sets of nearest-neighbor directions.

(a) Write down the partition function for a system of N such moments in equilibrium at temperature T. Neglect all non-magnetic degrees of freedom, i.e., just take into account the magnetic interaction energy $-\boldsymbol{\mu} \cdot \mathbf{H}$.

(b) Calculate the Helmholtz free energy of the system for arbitrary values of μ, H, and T.

(c) Calculate M, the magnitude of the total magnetization, for arbitrary values of μ, H, and T.

(d) Calculate the magnetic susceptibility $\chi = (\partial M/\partial H)_T$ for arbitrary values of μ, H, and T.

(e) Sketch the variation of M with H at fixed μ and T. You should pay special attention to getting the low-field and high-field limits qualitatively correct.

PHY 4523 Spring 2000 – Mid-Term Exam, Part 1

Note: (1) The questions below should not be used to practice for the upcoming mid-term. They are included only to show you the type of problem assigned last year. (2) Both questions are variants of problems assigned for homework during the first half of the Spring 2000 semester.

You have 50 minutes in which to attempt both problems, which carry equal weight. **To gain full credit you must explain your reasoning and show all working.** *You are allowed to consult Reif, Callen, and/or your lecture notes but no other materials.*

1. Consider a system consisting of two particles. Each particle can occupy any one of three states which have energies $-\varepsilon$, 0, and 2ε, respectively. Calculate the canonical partition function Z for this system in the case where the particles are ...

   (a) fermions;

   (b) bosons;

   (c) indistinguishable semiclassical particles obeying Maxwell-Boltzmann statistics.

2. Consider a system of N distinguishable, independent atoms. Every atom has two low-lying sets of energy levels: g_0 ground-state levels each having energy 0, plus g_1 excited levels each having energy ε. All other levels are sufficiently high in energy that they can be ignored.

   (a) Calculate the canonical partition function of this system in equilibrium at temperature T.

   (b) Determine the mean energy of the system at temperature T.

   (c) Calculate $C_v(T)$, the (total) heat capacity at constant volume.

   (d) The main feature of $C_v(T)$ is a "Schottky peak" at some temperature T_0. Derive an equation for T_0 in terms of ε, g_0, g_1, and constants. Simplify the equation as much as possible, but do not attempt to solve it algebraically.