# PHY 4604 Fall 2008 — Exam 1

1(a) Require

$$
\begin{aligned}
1 &= \int_{-\infty}^{+\infty} |\Psi(x,0)|^2\,dx \\
&= 4|A|^2\int_0^a x^2\,dx + |A|^2\int_a^{3a}(3a-x)^2\,dx \\
&= 4|A|^2\left[\tfrac{1}{3}x^3\right]_0^a + |A|^2\left[9a^2x - 3ax^2 + \tfrac{1}{3}x^3\right]_a^{3a} \\
&= |A|^2a^3\left(4\times\tfrac{1}{3} + 9\times 2 - 3\times 8 + \tfrac{1}{3}\times 26\right) \\
&= 4a^3|A|^2
\end{aligned}
$$

$$\Rightarrow \quad A = \frac{1}{2\sqrt{a^3}} \qquad \left(\text{or more generally } \frac{e^{i\theta}}{2\sqrt{a^3}},\ \theta \text{ real}\right)$$

(b)

$$
\begin{aligned}
P(x<a) &= \int_{-\infty}^{a} |\Psi(x,t)|^2\,dx \\
&= 4|A|^2\int_0^a x^2\,dx \\
&= \tfrac{4}{3}a^3 \times \frac{1}{4a^3} \\
&= \tfrac{1}{3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{+\infty} x\,|\Psi(x,0)|^2\,dx \\
&= 4|A|^2\int_0^a x^3\,dx + |A|^2\int_a^{3a} x(3a-x)^2\,dx \\
&= 4|A|^2\left[\tfrac{1}{4}x^4\right]_0^a + |A|^2\left[\tfrac{9}{2}a^2x^2 - 2ax^3 + \tfrac{1}{4}x^4\right]_a^{3a} \\
&= \frac{a^4}{4a^3}\left(1 + \tfrac{9}{2}\times 8 - 2\times 26 + \tfrac{1}{4}\times 80\right) \\
&= \tfrac{5}{4}a
\end{aligned}
$$

(d)

$$
\begin{aligned}
\hat{p}\,\Psi(x,0) &= -i\hbar\frac{\partial}{\partial x}\Psi(x,0) \\
&= \begin{cases} -2i\hbar A & \text{for } 0<x<a \\ i\hbar A & \text{for } a<x<3a \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \langle p\rangle &= \int_{-\infty}^{+\infty} \Psi^*(x,0)\,\hat{p}\,\Psi(x,0) \\
&= -4i\hbar|A|^2\int_0^a x\,dx + i\hbar|A|^2\int_a^{3a}(3a-x)\,dx \\
&= -4i\hbar|A|^2\left[\tfrac{1}{2}x^2\right]_0^a + i\hbar|A|^2\left[3ax - \tfrac{1}{2}x^2\right]_a^{3a} \\
&= -2i\hbar a^2|A|^2 + i\hbar a^2|A|^2\left(3\times 2 - \tfrac{1}{2}\times 8\right) \\
&= i\hbar a^2|A|^2(-2+6-4) \\
&= 0
\end{aligned}
$$

2(a) Can write $\quad \Psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$

with $\quad c_2 = \frac{i}{\sqrt{5}}, \quad c_7 = \frac{-2}{\sqrt{5}}$, all other $c_n = 0$.

$$\Rightarrow \qquad \Psi(x,T) = \sum_n c_n \psi_n(x) e^{-iE_nT/\hbar} \qquad (1)$$

where $\quad E_n = (n+\frac{1}{2})\hbar\omega$.

The Hamiltonian (energy operator) gives

$$\hat{H}\Psi(x,t) = \sum_n c_n e^{-iE_nT/\hbar} \hat{H}\psi_n(x)$$
$$= \sum_n c_n E_n e^{-iE_nT/\hbar} \psi_n(x)$$
$$\Rightarrow \qquad \langle H\rangle = \int_{-\infty}^{+\infty} \Psi^*(x,T)\hat{H}\Psi(x,T)\,dx$$
$$= \sum_n \sum_{n'} c_n^* c_{n'} E_{n'} e^{i(E_n-E_{n'})T/\hbar} \underbrace{\int_{-\infty}^{+\infty}\psi_n^*(x)\psi_{n'}(x)\,dx}_{\delta_{n,n'}}$$
$$= \sum_{n=0}^{\infty} |c_n|^2 E_n \qquad \text{independent of } T$$

In this particular case

$$\langle H\rangle = \frac{1}{5}\times\frac{5}{2}\hbar\omega + \frac{4}{5}\times\frac{15}{2}\hbar\omega$$
$$= \frac{13}{2}\hbar\omega$$

(b) The expansion of $\hat{p}^{2k+1} = \left[i\sqrt{2\hbar m\omega}(a_+ - a_-)\right]^{2k+1} \qquad (k = 0,1,2,\dots)$

contains terms proportional to $(a_+)^{2k+1}$, $(a_+)^{2k}(a_-)$, $(a_+)^{2k-1}(a_-)^2$, .... through $(a_-)^{2k+1}$.

When acting on $\psi_n$, $\hat{p}^{2k+1}$ gives terms proportional to $\psi_{n+2k+1}$, $\psi_{n+2k-1}$, $\psi_{n+2k-3}$, ..., $\psi_{n-2k+3}$, $\psi_{n-2k+1}$, and $\psi_{n-2k-1}$, but nothing proportional to $\psi_n$.

$\Rightarrow$ To get a nonzero expectation value

$$\langle p^{2k+1}\rangle = \int_{-\infty}^{+\infty} \Psi^*(x,T)\,\hat{p}^{2k+1}\,\Psi(x,T)\,dx$$

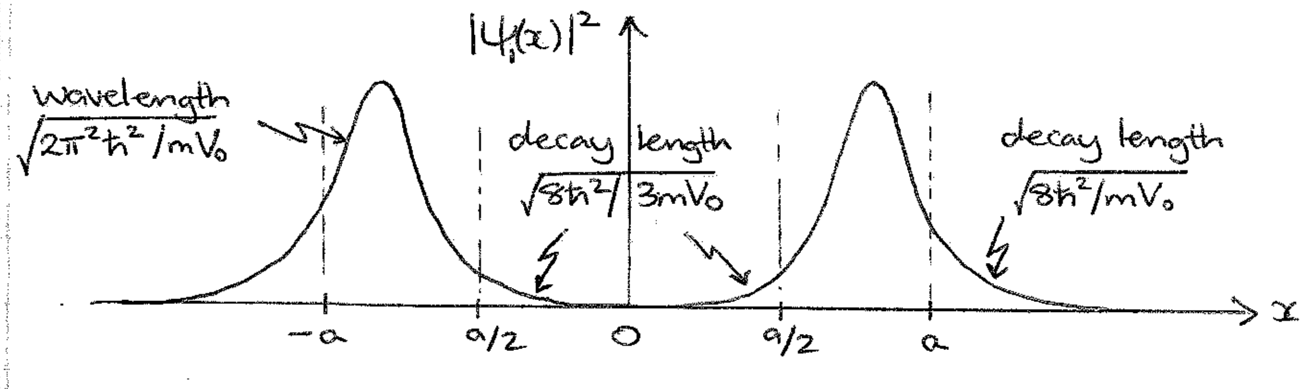
in the state given in Eq. (1), $k$ must be large enough that $\hat{p}^{2k+1}$ maps $\psi_2$ to a part proportional to $\psi_7$ and vice versa

i.e., smallest power of $p = 5$.

Note: $\quad \langle p^5\rangle = \frac{7!}{2!}(c_2^* c_7 + c_7^* c_2) = 7!\,\mathrm{Re}(c_2^* c_7)$

With $c_2 = \frac{i}{\sqrt{5}}$, $c_7 = -\frac{2}{\sqrt{5}}$, this is still zero. However, for general $c_2 \neq 0$ and $c_7 \neq 0$, the expectation value would be nonzero.

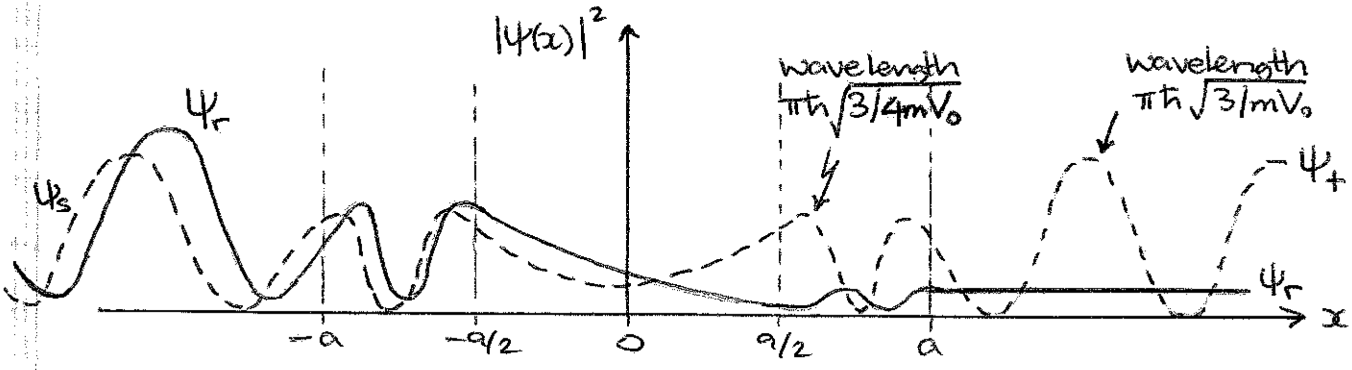
3(a) The first excited state wave function should have exactly one node. Since $V(-x) = V(x)$, we know that $\psi_1(-x) = -\psi_1(x)$.




Note: (i) Exponential decay length of $|\psi_1(x)|^2$ is shorter in region $|x| < a/2$ than in region $|x| > a$ $\Rightarrow$ peak in $|\psi_1(x)|^2$ lies closer to $|x| = a$.

(ii) Wavelength of $|\psi_1(x)|^2$ in region $a/2 < |x| < a$ is a bit more than $a$.

(b) Since there are two linearly independent states of energy $E = V_0/3 > V(\pm\infty)$, there are many possible answers. Below are illustrated the probability densities for $\psi_r$ (conventional scattering state with incident particle moving rightward from $x = -\infty$) and the symmetric state $\psi_s(x) = \psi_s(-x)$




The wavelength of $|\psi_r(x)|^2$ and $|\psi_s(x)|^2$ in the region $a/2 < |x| < a$ is half that in the region $|x| > a$, and is about 60% of the wavelength of $\psi_1(x)$ above in the region $a/2 < |x| < a$.

Note that $\psi_r$ has a nonzero probability current, so $|\psi_r(x)|^2 \neq 0$ anywhere $E > V(x)$, whereas $\psi_s$ has a zero probability current, so $|\psi_s(x)|^2 = 0$ once per wavelength.