PHY 4604 Fall 2008 — Exam 2

1(a) A stationary state $|\psi\rangle$ of energy E satisfies

$$\hat{H}|\psi\rangle \equiv \left(\frac{\hat{p}^2}{2m} + \hat{V}\right)|\psi\rangle = E|\psi\rangle.$$

The momentum-space wave function $\phi(p) = \langle p|\psi\rangle$ satisfies

$$\left[\frac{p^2}{2m} + V\left(i\hbar\frac{d}{dp}\right)\right]\phi = E\phi,$$

where in this case, $\quad V(x) = -Fx.$

$$\Rightarrow \qquad \frac{p^2}{2m}\phi - i\hbar F\frac{d\phi}{dp} = E\phi.$$

(b) From (a),

$$\frac{d\phi}{dp} = \frac{i}{\hbar F}\left(E - \frac{p^2}{2m}\right)\phi$$

$$\int\frac{d\phi}{\phi} = \frac{i}{\hbar F}\int\left(E - \frac{p^2}{2m}\right)dp$$

$$\phi(p) = \phi(0)\exp\left[\frac{i}{\hbar F}\left(Ep - \frac{p^3}{6m}\right)\right].$$

(c) The momentum probability density is

$$|\phi(p)|^2 = |\phi(0)|^2$$

which is independent of p.

2(a) A bound state must have energy $E < \min[V(\infty), V(-\infty)]$, i.e.,

$$E < \min(V_1, 0).$$

For a piecewise-constant potential, $\psi(x)$ must decay exponentially as $|x| \to \infty$:

$$\psi(x) = \begin{cases} Ae^{Kx} & \text{for } x<0 \quad K = \sqrt{-2mE}/\hbar \\ Be^{-K_1 x} & \text{for } x>0 \quad K_1 = \sqrt{2m(V_1-E)}/\hbar \end{cases}$$

(b) The wave function must be continuous at $x = 0$:

$$\Rightarrow \qquad B = A$$

The slope must undergo a jump due to the delta function in $V(x)$:

$$\Delta\psi'(0) = \frac{2m}{\hbar^2}(-aV_0)\psi(0)$$

$$-K_1 B - KA = -\frac{2maV_0}{\hbar^2}A$$

$$A = B \neq 0 \Rightarrow \qquad K + K_1 = \frac{2maV_0}{\hbar^2}$$

Canceling a factor of $\frac{\sqrt{2m}}{\hbar}$ on both sides of the equation,

$$\sqrt{-E} + \sqrt{V_1 - E} = 2\sqrt{|E_0|} \qquad (1)$$

where
$$E_0 = -\frac{ma^2V_0^2}{2\hbar^2} \quad \text{(as defined in the question).}$$

(c) If $V_1 > 0$, the band state must have energy $E < 0$.

$$(1) \Rightarrow \qquad 2\sqrt{|E_0|} = \sqrt{-E} + \sqrt{V_1 - E}$$
$$> \sqrt{V_1} \qquad (2)$$

If $V_1 < 0$, the band state must have energy $E < V_1$.

$$(1) \Rightarrow \qquad 2\sqrt{|E_0|} = \sqrt{-E} + \sqrt{V_1 - E}$$
$$> \sqrt{-V_1} \qquad (3)$$

Together, (2) and (3) imply that a band state exists only for
$$|V_1| < 4|E_0|.$$

(d) $(1)^2 \Rightarrow$
$$-E + 2\sqrt{-E(V_1-E)} + V_1 - E = 4|E_0|$$
$$2\sqrt{-E(V_1-E)} = 4|E_0| + 2E - V_1 \qquad (4)$$

$(4)^2 \Rightarrow$
$$-\cancel{4V_1E} + \cancel{4E^2} = 16|E_0|^2 + \cancel{4E^2} + V_1^2 + 16|E_0|E - 8|E_0|V_1 - \cancel{4V_1E}$$
$$16|E_0|E = -16|E_0|^2 + 8|E_0|V_1 - V_1^2$$
$$E = -|E_0| + \tfrac{1}{2}V_1 - \frac{V_1^2}{16|E_0|} = \frac{(E_0 + \frac{1}{4}V_1)^2}{E_0}$$

Check: For $V_1 = 0$ $\qquad E = E_0$ (as pointed out in the question).

3(a) The matrix elements of operator $\hat{\Omega}$ in the basis $\{|1\rangle, |2\rangle\}$ are
$$\Omega_{jk} = \langle j|\hat{\Omega}|k\rangle \qquad \text{for } j,k = 1,2.$$

Then
$$\hat{G} \leftrightarrow \gamma\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \hat{H} \leftrightarrow \epsilon\begin{pmatrix} 2 & 3i \\ -3i & 2 \end{pmatrix}.$$

(b) The eigenvalues E of $\hat{H}$ satisfy
$$0 = \det(\hat{H} - E\hat{I}) = (2\epsilon - E)^2 - |3i\epsilon|^2$$
$$= E^2 - 4\epsilon E - 5\epsilon^2 = (E - 5\epsilon)(E + \epsilon)$$

The eigenvalues are $E_1 = -\epsilon, \quad E_2 = 5\epsilon.$

The E_1 eigenvector satisfies

$$\begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ or } |E_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle).$$

Orthonormality $\Rightarrow \quad |E_2\rangle = \frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle)$

(c) If the initial state is

$$|\psi(0)\rangle = e^{i\pi/3}|1\rangle$$
$$= e^{i\pi/3}\frac{1}{\sqrt{2}}(|E_1\rangle - i|E_2\rangle)$$

then at time t the state is

$$|\psi(t)\rangle = e^{i\pi/3}\frac{1}{\sqrt{2}}\left(e^{i\epsilon t/\hbar}|E_1\rangle - ie^{-i5\epsilon t/\hbar}|E_2\rangle\right)$$
$$= e^{i\pi/3}\frac{1}{\sqrt{2}}\Big[e^{i\epsilon t/\hbar}\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$$
$$- ie^{-i5\epsilon t/\hbar}\frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle)\Big]$$
$$= e^{i\left(\frac{\pi}{3} - \frac{2\epsilon t}{\hbar}\right)}\Big[\frac{1}{2}\left(e^{i3\epsilon t/\hbar} + e^{-i3\epsilon t/\hbar}\right)|1\rangle$$
$$+ \frac{i}{2}\left(e^{i3\epsilon t/\hbar} - e^{-i3\epsilon t/\hbar}\right)|2\rangle\Big]$$
$$= e^{i\left(\frac{\pi}{3} - \frac{2\epsilon t}{\hbar}\right)}\left(\cos\frac{3\epsilon t}{\hbar}|1\rangle - i\sin\frac{3\epsilon t}{\hbar}|2\rangle\right)$$

The possible outcomes of a measurement of G are the eigenvalues:

$G = g_1 = \gamma$ with probability $P(g_1, t) = |\langle 1|\psi(t)\rangle|^2 = \cos^2\frac{3\epsilon t}{\hbar}$,

$G = g_2 = 2\gamma$ " " $P(g_2, t) = |\langle 2|\psi(t)\rangle|^2 = \sin^2\frac{3\epsilon t}{\hbar}$.

(d)

$$\langle G\rangle = \sum_n g_n P(g_n, t)$$
$$= \gamma\left(\cos^2\frac{3\epsilon t}{\hbar} + 2\sin^2\frac{3\epsilon t}{\hbar}\right)$$
$$= \gamma\left(1 + \sin^2\frac{3\epsilon t}{\hbar}\right)$$

which is maximized when

$$\sin^2\frac{3\epsilon t}{\hbar} = 1$$
$$\frac{3\epsilon t}{\hbar} = \left(n + \tfrac{1}{2}\right)\pi$$
$$t = \left(n + \tfrac{1}{2}\right)\frac{\pi\hbar}{3\epsilon}, \quad n = 0, 1, 2, \ldots$$