PHY 4604 Fall 2008 – Final Exam

1(a). A particle of momentum p has a de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}$$

Here,

$$\lambda = \frac{6.6\times10^{-34}\,\mathrm{J\,s}}{5.0\times10^{-24}\,\mathrm{kg\,m/s}}$$

$$= 1.3\times10^{-10}\,\mathrm{m}$$

Answer B

(b). A particle with orbital quantum number l has $2l+1$ allowed values of L_z and has $L^2 = l(l+1)\hbar^2$.

Here,

$$2l+1 = 7$$

$\Rightarrow$

$$l = 3$$

$$L^2 = 12\hbar^2$$

Answer D

2. We can write $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

where R and Y are separately normalized.

(a). Let

$$R(r) = A r e^{-r/a}.$$

Radial probability density is

$$\rho(r) = r^2|R(r)|^2$$

$$= |A|^2 r^4 e^{-2r/a}.$$

$$\Rightarrow \quad \frac{d\rho}{dr} = |A|^2(4r^3 - 2r^4/a)e^{-2r/a}.$$

The most probable r is the solution of

$$\frac{d\rho}{dr} = 0$$

$$4r^3 - 2r^4/a = 0$$

$$r = 2a.$$

(b). Let

$$Y(\theta,\phi) = B(1+\sqrt{6}\sin\theta\cos\phi).$$

Here,

$$1 \equiv \sqrt{4\pi}\,Y_0^0$$

and

$$\sin\theta\cos\phi = \tfrac{1}{2}\sin\theta(e^{i\phi}+e^{-i\phi})$$

$$\equiv \tfrac{1}{2}\sqrt{\tfrac{8\pi}{3}}\,(Y_1^{-1} - Y_1^{1}).$$

$$\Rightarrow \qquad Y(\theta,\phi) = B\left[\sqrt{4\pi}\,Y_0^0 + \sqrt{6}\sqrt{\tfrac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right)\right]$$
$$= \sqrt{4\pi}\,B\left(Y_0^0 + Y_1^{-1} - Y_1^1\right)$$
$$= \tfrac{1}{\sqrt{3}}\left(Y_0^0 + Y_1^{-1} - Y_1^1\right)$$

since the Y_l^m's are orthonormal.

Now, $\qquad \hat{L}^2 Y_l^m = l(l+1)\hbar^2 Y_l^m$

$$\Rightarrow \qquad \langle L^2 \rangle = \tfrac{1}{3}\left(0 + 2\hbar^2 + 2\hbar^2\right)$$
$$= \tfrac{4}{3}\hbar^2$$

3(a) The state given corresponds to a spinor in the S_z basis

$$\chi = \tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

In the same basis, the operator $\hat{S}_x$ is represented

$$S_x = \tfrac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\Rightarrow \qquad \langle S_x \rangle = \chi^{+} S_x \chi$$
$$= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \end{pmatrix}\tfrac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$= -\tfrac{2}{5}\hbar$$

(b) We must express the overall state $|\psi\rangle$ in terms of eigenstates of $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. Using the notation $|l, m_l; s, m_s\rangle$,

$$|\psi\rangle = \tfrac{1}{\sqrt{5}}\left(2|1,1;\tfrac{1}{2},\tfrac{1}{2}\rangle - |1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right).$$

From the $1 \otimes \tfrac{1}{2}$ Clebsch-Gordon table,

$$|1,1;\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle,$$
$$|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle,$$

using the notation $|j, m_j\rangle$ on the right-hand side of each equation.

$$\Rightarrow \qquad |\psi\rangle = \tfrac{2}{\sqrt{5}}|\tfrac{3}{2},\tfrac{3}{2}\rangle - \tfrac{1}{\sqrt{15}}|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{15}}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

In state $|j, m_j\rangle$, $\qquad |\vec{J}| = \sqrt{J^2} = \sqrt{j(j+1)}\,\hbar.$

In state $|\psi\rangle$, ...

$j = 3/2 \Rightarrow |\vec{J}| = \frac{\sqrt{15}}{2}\hbar$ with probability $\frac{4}{5} + \frac{1}{15} = \frac{13}{15}$,

$j = \frac{1}{2} \Rightarrow |\vec{J}| = \frac{\sqrt{3}}{2}\hbar$ " " $\frac{2}{15}$.

4(a). If the single-particle states are labeled $n = 0, 1, 2, \ldots$ then state n has energy

$$E_n = (n+1)^2 \frac{\hbar^2 \pi^2}{2ma^2}.$$

For distinguishable particles, can have any product state $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ with energy $E = \left[(n_1+1)^2 + (n_2+1)^2\right] E_0$. The three lowest-energy states are

$$\begin{array}{lll} \psi_0(x_1)\psi_0(x_2) & & E = 2E_0 \\ \psi_0(x_1)\psi_1(x_2) & \multirow{2}{*}{$\Big\}$ or any orthonormal linear combinations} & 5E_0 \\ \psi_1(x_1)\psi_0(x_2) & & 5E_0 \end{array}$$

(b). For identical bosons, the state must be symmetric under particle interchange. Since the spin state is symmetric, the spatial state must also be symmetric. The three lowest-energy states are

$$\begin{array}{ll} \psi_0(x_1)\psi_0(x_2) & E = 2E_0 \\ \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)\right] & 5E_0 \\ \psi_1(x_1)\psi_1(x_2) & 8E_0 \end{array}$$

(c). For identical fermions, the state must be antisymmetric under particle interchange. Since the spin state is symmetric, the spatial state must be antisymmetric. The three lowest-energy states are

$$\begin{array}{ll} \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)\right] & E = 5E_0 \\ \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2)\right] & 10E_0 \\ \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right] & 13E_0 \end{array}$$

(d). The Clebsch-Gordon tables show that a spin singlet state is symmetric under particle interchange for two integer spins but antisymmetric for two half-integer spins.

$\Rightarrow$ The distinguishable and bosonic spatial states would be unchanged, but the fermionic states would be those from (b).