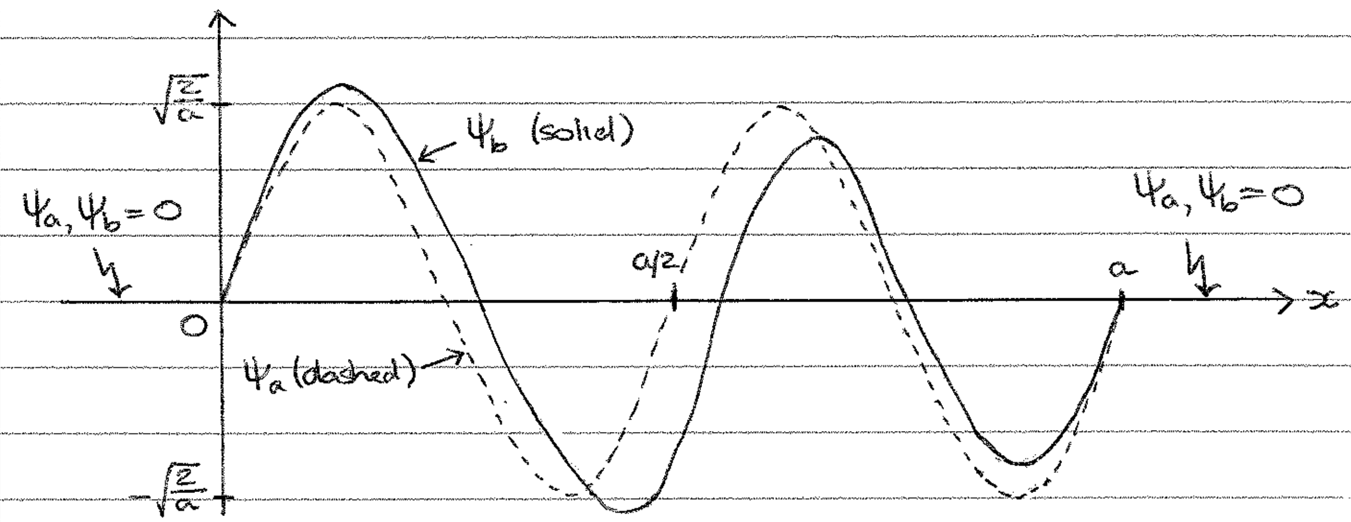
PHY 4604 Fall 2009 – Exam 1

1(a) $\psi_a(x) = \sqrt{\frac{2}{a}} \sin \frac{4\pi x}{a}$ for $0 \le x \le a$; $\psi_a(x) = 0$ for $x \le 0$ or $x \ge a$.
State has 3 nodes (at $x = \frac{a}{4}, \frac{a}{2}, \frac{3a}{4}$) and has energy $E_a = \frac{16\pi^2\hbar^2}{2ma^2}$.

(b) $\psi_b(x)$ also vanishes for $x \le 0$, $x \ge a$ and has 3 nodes at points $0 < x < a$.
Since $V_0 \ll E_a$, it is reasonable to assume that $\psi_b(x)$ has energy $E_b > V_0$.
Then $\psi_b''(x)/\psi_b(x) = \frac{2m}{\hbar^2}[V(x) - E_b]$ tells us that $\psi_b(x)$ is sinusoidal for $0<x<a$ with a longer wavelength for $0 < x < a/2$ than for $a/2 < x < a$.
Continuity of $\psi_b(x)$ and $\psi_b'(x)$ at $x = a/2$ (or analogy with the classical case) implies that $|\psi_b(x)|$ has a larger maximum value for $0<x<a/2$ than for $a/2<x<a$.
Finally, the area under $|\psi_b(x)|^2$ vs x must equal that under $|\psi_a(x)|^2$ vs x.



2. Can write $\quad \psi(x) = \frac{1}{\sqrt{5}}[2\psi_1(x) - \psi_2(x)]$
where $\quad \hat{H}\psi_n(x) = E_n\psi_n(x), \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

(a) Mean energy $\quad \langle E \rangle = \langle H \rangle = \int_{-\infty}^{+\infty} \psi^*(x)\,\hat{H}\,\psi(x)\,dx$

$$= \frac{1}{5}\int_{-\infty}^{+\infty} [2\psi_1^* - \psi_2^*][2E_1\psi_1 - E_2\psi_2]\,dx$$

Using the orthonormality property of the stationary states, i.e.,

$$\int_{-\infty}^{+\infty} \psi_m^*(x)\psi_n(x)\,dx = \delta_{m,n},$$

$$\Rightarrow \quad \langle E \rangle = \frac{1}{5}(4E_1 + E_2) = \frac{1}{5}(4\times 1 + 4)\frac{\pi^2\hbar^2}{2ma^2}$$

$$= \underline{\frac{4\pi^2\hbar^2}{5ma^2}}$$

(b) Mean-squared energy $\langle E^2\rangle = \langle H^2\rangle = \frac{1}{5}\int_{-\infty}^{+\infty}(2\psi_1^* - \psi_2^*)(2E_1^2\psi_1 - E_2^2\psi_2)\,dx$

$$= \frac{1}{5}(4E_1^2 + E_2^2) = \left(\frac{\pi^2\hbar^2}{ma^2}\right)^2$$

Energy uncertainty $\quad \underline{\underline{\sigma_E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \frac{3\pi^2\hbar^2}{5ma^2}}}$

(c) $\quad \text{Prob}(x < a/2) = \int_{-\infty}^{a/2}|\psi(x)|^2 dx = \frac{2}{5a}\int_0^{a/2}\left(4\sin^2\frac{\pi x}{a} + \sin^2\frac{2\pi x}{a} - 4\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right)dx$

$$= \frac{2}{5a}\left[\frac{4a}{\pi}\int_0^{\pi/2}\sin^2 y\,dy + \frac{a}{2\pi}\int_0^{\pi}\sin^2 y\,dy - \frac{8a}{\pi}\int_0^{\pi/2}(\cos y - \cos^3 y)\,dy\right]$$

using $\quad \sin y \sin 2y = \sin y \cdot 2\sin y\cos y = 2\cos y(1-\cos^2 y)$.

Applying the integrals given on the exam sheet,

$$\text{Prob}(x < a/2) = \frac{2}{5a}\left[\frac{4a}{\pi}\cdot\frac{\pi}{4} + \frac{a}{2\pi}\cdot\frac{\pi}{2} - \frac{8a}{\pi}\left(1 - 1 + \frac{1}{3}\right)\right]$$

$$\underline{\underline{= \frac{1}{2} - \frac{16}{15\pi}}}$$

3(a) $\quad \underline{\underline{\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right) = \frac{1}{\sqrt{2}}\left[\psi_1(x) + \psi_2(x)e^{-i\omega t}\right]e^{-\frac{3}{2}i\omega t}}}$

(b) Use $\quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad$ where $\quad a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

$\Rightarrow$ Mean $\quad \langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2}\int_{-\infty}^{+\infty}(\psi_1^* + \psi_2^* e^{i\omega t})(\sqrt{2}\psi_2 + \sqrt{3}\psi_3 e^{-i\omega t} + \psi_0 + \sqrt{2}\psi_1 e^{-i\omega t})\,dx$

$$\underline{\underline{= \sqrt{\frac{\hbar}{m\omega}}\cos\omega t}} \qquad \underline{\underline{\text{Vanishes for } t = (n+\tfrac{1}{2})\tfrac{\pi}{\omega} \text{ (n integer)}}}$$

Above, have used the orthonormality property of the stationary states.

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\,\frac{1}{2}\int_{-\infty}^{+\infty}(\psi_1^* + \psi_2^* e^{i\omega t})(\sqrt{6}\psi_3 + \sqrt{12}\psi_4 e^{-i\omega t} + \psi_1 + 2\psi_2 e^{-i\omega t} + 2\psi_1 + 3\psi_2 e^{-i\omega t} + 0 + \sqrt{2}\psi_0 e^{-i\omega t})\,dx$$

$$= \frac{2\hbar}{m\omega}$$

$\Rightarrow$ Std. deviation $\quad \underline{\underline{\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{(2-\cos^2\omega t)\frac{\hbar}{m\omega}}}}$

(c) Use $\quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$

$\Rightarrow$ Mean $\quad \langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\frac{1}{2}\int_{-\infty}^{+\infty}(\psi_1 + \psi_2^* e^{i\omega t})(\sqrt{2}\psi_2 + \sqrt{3}\psi_3 e^{-i\omega t} - \psi_0 - \sqrt{2}\psi_1 e^{-i\omega t})\,dx$

$$\underline{\underline{= -\sqrt{\hbar m\omega}\sin\omega t}} \qquad \underline{\underline{\text{Vanishes for } t = n\pi/\omega \text{ (n integer)}}}$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\,\frac{1}{2}\int_{-\infty}^{+\infty}(\psi_1^* + \psi_2^* e^{i\omega t})(\sqrt{6}\psi_3 + \sqrt{12}\psi_4 e^{-i\omega t} - \psi_1 - 2\psi_2 e^{-i\omega t} - 2\psi_1 - 3\psi_2 + 0 + \sqrt{2}\psi_0 e^{-i\omega t})\,dx$$

$$= 2\hbar m\omega$$

$\Rightarrow$ Std. deviation $\quad \underline{\underline{\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{(2-\sin^2\omega t)\hbar m\omega}}}$