PHY 4604 Fall 2009 — Exam 2

DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Instructions: Attempt both questions. The maximum possible credit for each part of each question is shown in square brackets. Please try to write your solution neatly and legibly.

You will receive credit only for knowledge and understanding that you demonstrate in your written solutions. It is in your best interest to write down something relevant for every question, even if you can't provide a complete answer. To maximize your score, you should briefly explain your reasoning and show all working. Give all final algebraic answers in terms of variables defined in the problem and $\hbar$ (the reduced Planck constant).

During this exam, you may use two formula sheets. You are not permitted (a) to consult any other books, notes, or papers, (b) to use any electronic device, or (c) to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

Print your name where indicated below, and sign to confirm that you have read and understood these instructions. Please do not write anything else below the line.

Name (printed): ____________________ Signature: ____________________

Question	Score
1	______
2	______
Total	

The following may be useful:

$$\coth x = (e^x + e^{-x})/(e^x - e^{-x}) \qquad\qquad \coth x \simeq 1 \text{ for } x \gg 1$$

$$x \coth x \simeq 1 \text{ for } |x| \ll 1 \qquad\qquad x \coth x > \max(1, |x|) \text{ for all } x \neq 0$$

1. [50 points total] A particle of mass m moves in one dimension under the potential $V(x) = -(\hbar^2 K_0/m)\,\delta(x)$ for $|x| < a$; $V(x) = \infty$ for $|x| > a$. Here, a and K_0 are positive real numbers with dimensions of length and $(\text{length})^{-1}$, respectively. This question focuses on the existence (or nonexistence) of a stationary state $\psi_1(x)$ of negative energy. If it exists, such a state is necessarily the ground state. Its energy can be written $E_1 = -\hbar^2 K^2/(2m)$, where K is real and positive.

   (a) [10 points] Write down the general form of such a negative-energy $\psi_1(x)$. Include only terms whose amplitude may be nonzero in a normalized state.

   (b) [8 points] Use the fact that the potential satisfies $V(-x) = V(x)$ to re-express the wave function in terms of just two unknown amplitudes.

   (c) [12 points] Apply the appropriate boundary conditions at $x = 0$ and $x = a$ to obtain two equations connecting K and the two amplitudes referred to in (b).

   (d) [8 points] Show that the equations you obtained in (c) imply $Ka \coth Ka = K_0 a$.

   (e) [6 points] Give a condition involving $K_0 a$ for the existence of a negative-energy bound state.

   (f) [6 points] Show that for $a \to \infty$ (with K_0 fixed), E_1 approaches the energy $-\hbar^2 K_0^2/(2m)$ of the bound state of the pure delta-function potential $V(x) = -(\hbar^2 K_0/m)\,\delta(x)$.

2. [50 points total] A quantum-mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors $\{|1\rangle, |2\rangle\}$. In this basis, the Hamiltonian $\hat{H}$ and another observable operator $\hat{A}$ have matrix representations

$$\hat{H} = \epsilon \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \hat{A} = \alpha \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

   Here, ϵ and α are positive, real quantities.

   (a) [8 points] Express $\hat{H}$ and $\hat{A}$ in outer-product form in the basis $\{|1\rangle, |2\rangle\}$.

   (b) [12 points] Find the eigenvalues E_1 and E_2 of $\hat{H}$ (choosing $E_1 < E_2$) and express the corresponding eigenkets $|E_1\rangle$ and $|E_2\rangle$ as linear combinations of $|1\rangle$ and $|2\rangle$.

   (c) [10 points] At time zero, the system is described by a state vector $(|1\rangle + i|2\rangle)/\sqrt{2}$. Express the state vector at time $t > 0$ as a linear combination of $|1\rangle$ and $|2\rangle$.

   (d) [20 points] What are the possible outcomes of a measurement of A performed at time $t > 0$ on the state defined in part (c)? Give the probability of each outcome.