PHY 4604 Fall 2009 - Exam 2

1(a) The general form of the wave function is

$$\psi_1(x) = \begin{cases} A_1 e^{Kx} + B_1 e^{-Kx} & \text{for } -a \leq x \leq 0 \\ A_2 e^{Kx} + B_2 e^{-Kx} & \text{for } 0 \leq x \leq a \\ 0 & \text{for } |x| \geq a \end{cases}$$

(b) Since $V(-x) = V(x)$, the ground state must be even, i.e., $\psi_1(-x) = \psi_1(x)$.
Then

$$\psi_1(x) = \begin{cases} A_1 e^{Kx} + B_1 e^{-Kx} & \text{for } -a \leq x \leq 0 \\ B_1 e^{Kx} + A_1 e^{-Kx} & \text{for } 0 \leq x \leq a \\ 0 & \text{for } |x| \geq a. \end{cases}$$

(c) Require continuity of $\psi_1(x)$ at $x = 0$ and $x = a$:

$x = 0$: $\quad A_1 + B_1 = B_1 + A_1 \quad$ automatically fulfilled

$x = a$: $\quad \underline{\underline{B_1 e^{Ka} + A_1 e^{-Ka} = 0}} \quad$ condition ①

Also require $\quad \Delta\psi_1'(0) = -\frac{2m}{\hbar^2} \frac{\hbar^2 K_0}{m} \psi_1(0)$

$$K(B_1 - A_1) - K(A_1 - B_1) = -2K_0(A_1 + B_1)$$

$$\underline{\underline{K(A_1 - B_1) = K_0(A_1 + B_1)}} \quad \text{condition ②}$$

(d) ① $\Rightarrow$ $\quad B_1 = -A_1 e^{-2Ka} \quad$ ③

②,③ $\Rightarrow$ $\quad K(1 + e^{-2Ka}) = K_0(1 - e^{-2Ka})$

$\times\, a e^{Ka}$ $\Rightarrow$ $\quad \underline{\underline{Ka \coth Ka = K_0 a}} \quad$ ④

(e) We need to find a solution $K > 0$ to ④. Since $Ka \coth Ka > 1$ for all $K > 0$, such a solution exists only if

$$\underline{\underline{K_0 a > 1}}$$

(f) For $a \to \infty$, the solution to ④ becomes $K \to K_0$

$$\Rightarrow \quad \underline{\underline{E_1 \to -\frac{\hbar^2 K_0^2}{2m}}}$$

which, as noted in the question, is the bound state energy of $V(x) = -\frac{\hbar^2 K_0}{m}\delta(x)$.

2(a) From a matrix representation Ω_{mn} in basis $\{|\chi_n\rangle\}$ we can construct

$$\hat{\Omega} = \sum_{m,n} \Omega_{mn} |\chi_m\rangle\langle\chi_n|$$

Here,

$$\hat{H} = \epsilon\left(|1\rangle\langle 1| + |2\rangle\langle 2| + 2|1\rangle\langle 2| + 2|2\rangle\langle 1|\right)$$

$$\hat{A} = i\alpha\left(|1\rangle\langle 2| - |2\rangle\langle 1|\right)$$

(b) The E_j satisfy $\quad 0 = \det(\hat{H} - E_j\hat{I}) = (\epsilon - E_j)^2 - (2\epsilon)^2$

$$\Rightarrow \quad E_1 = -\epsilon, \quad E_2 = 3\epsilon$$

E_1 eigenvector: $\quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

or $\quad |E_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right)$

each unique up to an overall complex phase

By orthonormality,

$$|E_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

(c) From (b), $\quad |1\rangle = \frac{1}{\sqrt{2}}\left(|E_1\rangle + |E_2\rangle\right) \quad$ and $\quad |2\rangle = \frac{1}{\sqrt{2}}\left(|E_2\rangle - |E_1\rangle\right)$

$$\Rightarrow \quad |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right) = \frac{1-i}{2}|E_1\rangle + \frac{1+i}{2}|E_2\rangle$$

Now, $|E_j\rangle$ at time 0 evolves to $e^{-iE_jt/\hbar}$ at time t.

$$\Rightarrow \quad |\psi(t)\rangle = \frac{1-i}{2}e^{-iE_1t/\hbar}|E_1\rangle + \frac{1+i}{2}e^{-iE_2t/\hbar}|E_2\rangle$$

$$= \frac{1-i}{2\sqrt{2}}e^{-iE_1t/\hbar}\left(|1\rangle - |2\rangle\right) + \frac{1+i}{2\sqrt{2}}e^{-iE_2t/\hbar}\left(|1\rangle + |2\rangle\right)$$

(d) A measurement of A will yield one of the eigenvalues a_j of $\hat{A}$, where

$$0 = \det(\hat{A} - a_j\hat{I}) = (-a_j)^2 - (i\alpha)(-i\alpha)$$

$$\Rightarrow \quad a_1 = -\alpha, \quad a_2 = \alpha \quad \text{are the possible values of A.}$$

a_1 eigenvector: $\quad \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

or $\quad |A = -\alpha\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right) = |\psi(0)\rangle$

$$\Rightarrow \langle A = -\alpha|\psi(t)\rangle = \frac{1-i}{2\sqrt{2}}e^{-iE_1t/\hbar}\frac{1}{\sqrt{2}}(1+i) + \frac{1+i}{2\sqrt{2}}e^{-iE_2t/\hbar}\frac{1}{\sqrt{2}}(1-i)$$

$$= \frac{1}{2}\left(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right) = \frac{1}{2}e^{-i\epsilon t/\hbar}\left(e^{+i2\epsilon t/\hbar} + e^{-i2\epsilon t/\hbar}\right)$$

$$= e^{-i\epsilon t/\hbar}\cos\frac{2\epsilon t}{\hbar}$$

$$\text{Prob}(A = -\alpha) = |\langle A = -\alpha|\psi(t)\rangle|^2 = \cos^2\frac{2\epsilon t}{\hbar}$$

$$\text{Prob}(A = \alpha) = 1 - \text{Prob}(A = -\alpha) = \sin^2\frac{2\epsilon t}{\hbar}$$