PHY 4604 Fall 2009 – Final Exam

1(a) A state with azimuthal quantum number l has angular momentum of magnitude $|\vec{L}| = \sqrt{l(l+1)}\,\hbar$ and $2l+1$ allowed values of L_z. Thus, a magnitude $|\vec{L}| = \sqrt{42}\,\hbar$ corresponds to $l = 6$ and 13 allowed L_z values.

Answer: E.

(b) Since $\hat{H} = \hat{H}_x + \hat{H}_y$ where $\hat{H}_x, \hat{H}_y$ describe separate one-dimensional harmonic oscillators, the stationary states have energies

$$\left.\begin{aligned} E &= (n_x + \tfrac{1}{2})\hbar\omega + (n_y + \tfrac{1}{2})\hbar\omega \\ &= (n_x + n_y + 1)\hbar\omega \end{aligned}\right\} \begin{array}{l} n_x, n_y \text{ are non-} \\ \text{negative integers} \end{array}$$

Answer: B

2(a) We know $n = 2$ and $l = 1$, but we don't know the L_z state of the atom. Since the energy does not depend on the magnetic quantum number m, the most general form of the stationary-state wave function is

$$\underline{\underline{\psi(r,\theta,\phi) = R_{21}(r) \sum_{m=-1}^{1} c_m Y_1^m(\theta,\phi)}}$$

where c_1, c_0, c_{-1} are complex numbers satisfying

$$\underline{\underline{\sum_{m=-1}^{m} |c_m|^2 = 1.}}$$

(b) Since the Y_1^m component of the wave function corresponds to $L_z = m\hbar$, the possible results are $\underline{\underline{L_z = 0, \pm\hbar}}$

(c) The radial probability density is

$$P(r) = r^2 R_{21}(r)^2 = r^2 \cdot \frac{r^2}{24a_0^5} e^{-r/a_0}$$

The most probable electron-nucleus distance r_p satisfies

$$0 = \left.\frac{dP}{dr}\right|_{r=r_p} = \frac{1}{24a_0^5}\left(4r^3 - \frac{r}{a_0}\right)e^{-r/a_0}$$

$$\Rightarrow \quad \underline{\underline{r_p = 4a_0}}$$

(d) $$\langle V \rangle = -e^2 \left\langle \frac{1}{r} \right\rangle \quad (\text{CGS}) \quad \text{or} \quad -\frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{r} \right\rangle \quad (\text{SI})$$

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty \frac{1}{r} P(r)\,dr = \frac{1}{24a_0^5} \int_0^\infty r^3 e^{-r/a_0}\,dr$$

$$= \frac{1}{24a_0} \underbrace{\int_0^\infty x^3 e^{-x}\,dx}_{6!\text{ (from formula sheet)}} = \frac{1}{4a_0}$$

$$\Rightarrow \quad \langle V \rangle = -\frac{e^2}{4a_0} \ (\text{CGS}) \text{ or } -\frac{e^2}{16\pi\epsilon_0 a_0} \ (\text{SI})$$

Total energy $E = \left\langle \frac{p^2}{2m} \right\rangle + \langle V \rangle$

$$= -\frac{e^2}{2n^2 a_0} \ (\text{CGS}) \text{ or } -\frac{e^2}{8\pi n^2 \epsilon_0 a_0} \ (\text{SI})$$

$$\Rightarrow \quad \left\langle \frac{p^2}{2m} \right\rangle = E - \langle V \rangle \quad (\text{using } n=2, \langle V \rangle \text{ from above})$$

$$= \frac{e^2}{8a_0} \ (\text{CGS}) \text{ or } \frac{e^2}{32\pi\epsilon_0 a_0} \ (\text{SI})$$

3(a) Hamiltonian $\hat{H} = \hat{H}_0 - \gamma B \hat{S}_z$

where $\hat{H}_0$ is the spatial part, which gives a constant energy E_0.
Thus, the $\hat{S}_z$ eigenkets are stationary spin kets, with energy

$$E = E_0 \mp \tfrac{1}{2}\gamma B \hbar \qquad (- \text{ for } |\uparrow\rangle, + \text{ for } |\downarrow\rangle)$$

and the initial state evolves to

$$|\chi(t)\rangle = \frac{e^{-iE_0 t/\hbar}}{5}\left(3e^{i\gamma Bt/2}|\uparrow\rangle + 4i\,e^{-i\gamma Bt/2}\right)$$

(b) $$\langle S_j \rangle = \langle \chi | \hat{S}_j | \chi \rangle = \chi^\dagger \frac{\hbar}{2} \sigma_j \chi$$

where $\chi(t) = \dfrac{e^{-iE_0 t/\hbar}}{5} \begin{pmatrix} 3\alpha \\ 4i\alpha^* \end{pmatrix}$ with $\alpha = e^{i\gamma Bt/2}$

and σ_j $(j = x, y, z)$ is a Pauli matrix.

$$\langle S_x \rangle = \tfrac{1}{5}(3\alpha^* \ \ -4i\alpha)\, \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tfrac{1}{5} \begin{pmatrix} 3\alpha \\ 4i\alpha^* \end{pmatrix}$$

$$= \frac{\hbar}{50}\left(12i\,\alpha^{*2} - 12i\,\alpha^2\right)$$

$$= \frac{12}{25}\hbar \sin \gamma Bt$$

$$\langle S_y \rangle = \tfrac{1}{5}(3\alpha^* \ \ -4i\alpha)\, \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tfrac{1}{5} \begin{pmatrix} 3\alpha \\ 4i\alpha^* \end{pmatrix}$$

$$= \frac{\hbar}{50}\left(12\alpha^{*2} + 12\alpha^2\right)$$

$$= \frac{12}{25}\hbar \cos \gamma Bt$$

$$\langle S_z\rangle = \frac{1}{5}\left(3\alpha^* \quad -4i\alpha\right)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{5}\begin{pmatrix}3\alpha\\4i\alpha^*\end{pmatrix}$$
$$= \frac{\hbar}{50}\left(9|\alpha|^2-16|\alpha|^2\right)$$
$$= -\frac{7\hbar}{50}$$

4(a) Since the particles have spin-½, they are fermions, so their wave function must satisfy

$$\psi(x_1,m_1,x_2,m_2) = -\psi(x_2,m_2,x_1,m_1).$$

The spatial part is symmetric under exchange, so the spin part must be antisymmetric. There is only one antisymmetric state of two spin-½ s:

$$|\chi\rangle = \frac{e^{i\theta}}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right), \quad \theta \text{ real}$$

(b) The Hamiltonian doesn't depend on the spin. The spatial part of the wave function can be written

$$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_0(x_2)+\psi_0(x_2)\psi_1(x_1)\right]$$

where $\psi_n(x)$ is the single-particle stationary state of the harmonic potential with energy $(n+\frac{1}{2})\hbar\omega$. Thus, the energy of the state in Eq. (1) is

$$E = \tfrac{3}{2}\hbar\omega + \tfrac{1}{2}\hbar\omega$$
$$= 2\hbar\omega$$

(c) Another possible state of energy $2\hbar\omega$ is

$$\psi(x_1,m_1,x_2,m_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_0(x_2)-\psi_0(x_1)\psi_1(x_2)\right]\langle m_1,m_2|\chi'\rangle$$
$$= \sqrt{\frac{2}{\pi}}\,\frac{x_1-x_2}{a^2}\exp\!\left(-\frac{x_1^2+x_2^2}{2a^2}\right)\langle m_1,m_2|\chi'\rangle$$

where $|\chi'\rangle$ is any symmetric spin state, i.e., any

$$|\chi'\rangle = c_1|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{c_0}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|-\tfrac{1}{2},\tfrac{1}{2}\rangle\right) + c_{-1}|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

with

$$|c_1|^2+|c_0|^2+|c_{-1}|^2 = 1$$