PHY 4604 Fall 2010 – Exam 1

1(a) Any state of the harmonic oscillator can be written

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \psi_n(x) \quad \text{with } E_n = (n+\tfrac{1}{2})\hbar\omega.$$

Examination of the given initial state $\Psi(x,0)$ tells us that

$$c_1 = \frac{i}{\sqrt{3}}, \quad c_3 = -\sqrt{\frac{2}{3}}, \quad c_n = 0 \text{ for } n \neq 1,3.$$

$$\Rightarrow \quad \underline{\underline{\Psi(x,T) = \frac{i}{\sqrt{3}} e^{-3i\omega T/2} \psi_1(x) - \sqrt{\frac{2}{3}} e^{-7i\omega T/2} \psi_3(x)}}$$

(b) An energy measurement at time t gives the result E_n with probability $|c_n e^{-iE_n t/\hbar}|^2 = |c_n|^2$ (independent of time).

In this particular example, the possible results are

$$\underline{\underline{E_1 = \tfrac{3}{2}\hbar\omega \quad \text{with probability} = \tfrac{1}{3}}}$$

$$\underline{\underline{E_2 = \tfrac{7}{2}\hbar\omega \quad \text{with probability} = \tfrac{2}{3}}}$$

(c)

$$\hat{p}\,\Psi(x,T) = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)\left(\frac{i}{\sqrt{3}} e^{-3i\omega T/2}\psi_1 - \sqrt{\frac{2}{3}} e^{-7i\omega T/2}\psi_3\right)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\Big[\frac{i}{\sqrt{3}} e^{-3i\omega T/2}(\sqrt{2}\psi_2 - \psi_0) - \sqrt{\frac{2}{3}} e^{-7i\omega T/2}(\sqrt{4}\psi_4 - \sqrt{3}\psi_2)\Big]$$

using $a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$ and $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$. The stationary states ψ_0, ψ_2 and ψ_4 entering $\hat{p}\,\Psi(x,T)$ are orthogonal to ψ_1 and ψ_3 in $\Psi(x,T)$.

$$\Rightarrow \quad \langle p \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,T)\, \hat{p}\, \Psi(x,T)\, dx$$

$$\underline{\underline{= 0}} \quad \text{using } \int_{-\infty}^{+\infty} \psi_n^*(x)\psi_m(x)\,dx = \delta_{n,m}.$$

$$\hat{p}^2\Psi(x,T) = -\frac{\hbar m\omega}{2}\Big[\frac{i}{\sqrt{3}} e^{-3i\omega T/2}(\sqrt{6}\psi_3 - 2\psi_1 - \psi_1 + 0) - \sqrt{\frac{2}{3}} e^{-7i\omega T/2}(\sqrt{20}\psi_5 - 4\psi_3 - 3\psi_3 + \sqrt{6}\psi_1)\Big]$$

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,T)\, \hat{p}^2\, \Psi(x,T)\, dx$$

$$= -\frac{\hbar m\omega}{2}\Big[\Big(-\frac{i}{\sqrt{3}} e^{3i\omega T/2}\Big)\Big(-\sqrt{3}\,i e^{-3i\omega T/2} - 2e^{-7i\omega T/2}\Big) + \Big(-\sqrt{\frac{2}{3}} e^{7i\omega T/2}\Big)\Big(\sqrt{2}\,i e^{-3i\omega T/2} + 7\sqrt{\frac{2}{3}} e^{-7i\omega T/2}\Big)\Big]$$

$$\underline{\underline{= \frac{1}{6}\hbar m\omega\left[17 - 4\sqrt{3}\sin(2\omega T)\right]}}$$

Std. dev. $\underline{\underline{\sigma_p = \langle p^2 \rangle - \langle p \rangle^2 = \left\{\frac{1}{6}\hbar m\omega\left[17 - 4\sqrt{3}\sin(2\omega T)\right]\right\}^{\frac{1}{2}}}}$

2(a) For $x<0$, $\underline{\psi(x) = A_1 e^{Kx}}$ so that ψ remains finite as $x \to -\infty$.

where $K = \dfrac{\sqrt{2m(V_0-E)}}{\hbar} = \dfrac{\sqrt{mV_0}}{\hbar}$ $\quad$ Exponential decay length

$\underline{\underline{= \dfrac{1}{a}}}$ $\quad$ $\underline{\underline{l = \dfrac{1}{K} = a}}$

(b) For $x>0$, $\underline{\psi(x) = A_2 e^{ikx} + B_2 e^{-ikx}}$

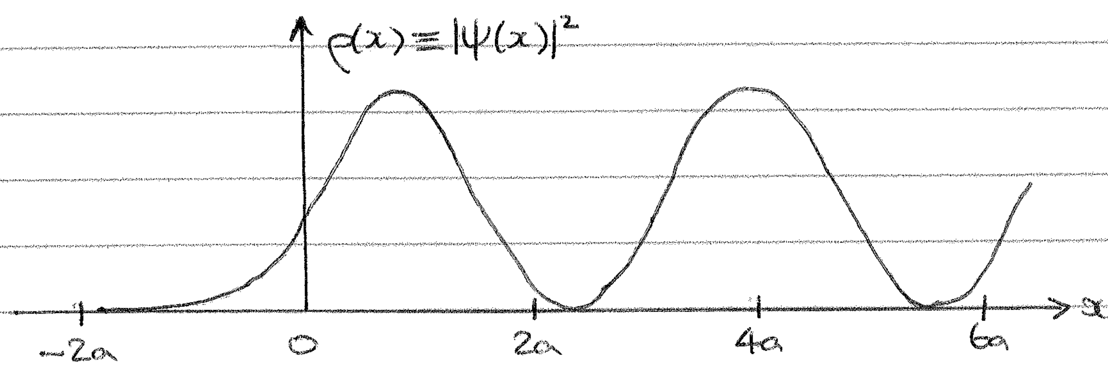
where $k = \dfrac{\sqrt{2mE}}{\hbar} = \dfrac{\sqrt{mV_0}}{\hbar}$ $\quad$ Wavelength

$\underline{\underline{= \dfrac{1}{a}}}$ $\quad$ $\underline{\underline{\lambda = \dfrac{2\pi}{k} = 2\pi a}}$

(c) For $x<0$, $|\psi(x)|^2 = |A_1|^2 e^{2Kx}$ decreases by a factor of $e^{-1} \simeq \frac{1}{3}$ when $|x|$ increases by $\frac{l}{2} = \frac{a}{2}$.

For $x>0$, $j = \frac{\hbar k}{m}(|A_2|^2 - |B_2|^2)$ must vanish just as for $x<0$.

$\Rightarrow |\psi(x)|^2$ oscillates with wavelength $\frac{\lambda}{2} = \pi a$ between 0 and $4|A_2|^2$.



3. The property $\Psi(x,0) = -\Psi(-x,0)$ exhibited by the initial wave function greatly simplifies the answers to (a)–(e).

(a)
$$\langle x \rangle = \int_{-\infty}^{+\infty} \underset{\text{odd}}{\Psi^*(x,0)}\ \underset{\text{odd}}{x}\ \underset{\text{odd}}{\Psi(x,0)}\, dx$$
$$\underline{\underline{= 0}} \qquad \text{since the integrand is odd}$$

(b)
$$\hat{p}\,\Psi(x,0) = -i\hbar \frac{d\Psi(x,0)}{dt}$$
$$= -i\hbar\sqrt{\frac{105}{16L^7}}\,(L^2 - 3x^2)$$
$$= \hat{p}\,\Psi(-x,0)$$
$$\langle p \rangle = \int_{-\infty}^{+\infty} \underset{\text{odd}}{\Psi^*(x,0)}\ \underset{\text{even}}{\hat{p}\,\Psi(x,0)}\, dx = 0.$$

(c)
$$\hat{p}^2\Psi(x,0) = -i\hbar\frac{d}{dx}\left[\hat{p}\Psi(x,0)\right]$$
$$= 6\hbar^2\sqrt{\frac{105}{16L^7}}\,x$$
$$\langle p^2\rangle = \int_{-\infty}^{+\infty}\Psi^*(x,0)\,\hat{p}^2\Psi(x,0)\,dx$$
$$= \frac{6\times105}{16}\frac{\hbar^2}{L^7}\int_{-L}^{L}(L^2x^2 - x^4)\,dx$$
$$= \frac{6\times105}{16}\frac{\hbar^2}{L^7}\left(\frac{2}{3}L^5 - \frac{2}{5}L^5\right)$$
$$= \frac{21\hbar^2}{2L^2}$$

Uncertainty $\quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$
$$= \sqrt{\frac{21}{2}}\,\frac{\hbar}{L}$$

(d) Can write $\quad \Psi(x,0) = \sum_{n=1}^{\infty} c_n\psi_n(x)$

where $\quad c_n = \int_{-\infty}^{+\infty}\psi_n^*(x)\,\Psi(x,0)\,dx$

with
$$\left.\begin{aligned}\psi_{n\,\text{odd}}(x) &= \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{2L} = \psi_n(-x)\\ \psi_{n\,\text{even}}(x) &= \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{2L} = -\psi_n(-x)\\ \text{having energy}\quad E_n &= \frac{n^2\pi^2\hbar^2}{8mL^2}\end{aligned}\right\}\ \text{standard stationary states with } a \to 2L$$

Since $\Psi(x,0) = -\Psi(-x,0)$, must have $c_n = 0$ for all odd n.

However, there is no reason for c_n to vanish for any even n.

$\Rightarrow$ Smallest value that can be obtained in an energy measurement is
$$E_2 = \frac{\pi^2\hbar^2}{2mL^2}$$

(e) From (d), $\quad \Psi(x,t) = \sum_{n\ \text{even}} c_n e^{-iE_nt/\hbar}\psi_n(x)$

This means $\quad \Psi(x,t) = -\Psi(-x,t) \quad \Rightarrow \langle x\rangle = 0$

and $\quad \hat{p}\Psi(x,t) = +\hat{p}\Psi(-x,t) \quad \Rightarrow \langle p\rangle = 0$

and prob. of measuring $E_n = |c_n|^2 = 0$ for n odd $\Rightarrow E_{min} = E_2$

Also, $\langle H\rangle = \langle \frac{p^2}{2m} + V\rangle$ remains constant in any QM state.

In this problem $\langle V\rangle = 0$, so $\langle p^2\rangle$ remains constant in time

$\Rightarrow$ Answers to (a)–(d) must be the same at $t = T$ as at $t = 0$.