PHY 4604 Fall 2010 - Exam 2

1(a) In regions (1) $x < -d$, (2) $|x| \leq d$, and (3) $x > d$, $V(x) = 0 > E$, so

$$\psi_0(x) = A_j e^{Kx} + B_j e^{-Kx} \qquad j = 1, 2, \text{or } 3.$$

Enforcing $\psi_0(x) \to 0$ as $|x| \to \infty$ and $\psi_0(x) = \psi_0(-x)$ leads to

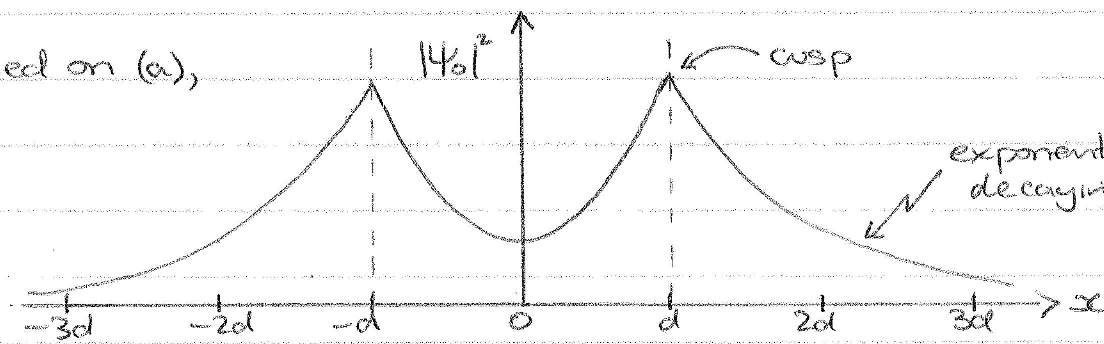
$$\underline{\underline{\psi_0(x) = \begin{cases} A_1 e^{Kx} & \text{for } x < -d \\ B_2 (e^{Kx} + e^{-Kx}) & \text{for } |x| < d \\ A_1 e^{-Kx} & \text{for } x > d. \end{cases}}}$$

(b) Apply boundary conditions at $x = d$ (no additional info from $x = -d$):

$$\Delta\psi_0 = 0 \quad \Rightarrow \quad \underline{\underline{A_1 e^{-Kd} = B_2 (e^{Kd} + e^{-Kd})}} \qquad ①$$

$$\Delta\psi_0' = \frac{2m}{\hbar^2}\left(-\frac{\hbar^2 v}{2md}\right)\psi_0 \quad \Rightarrow \quad \underline{\underline{-KA_1 e^{-Kd} = KB_2 (e^{Kd} - e^{-Kd}) - \frac{v}{d} A_1 e^{-Kd}}} \qquad ②$$

(c) Based on (a),

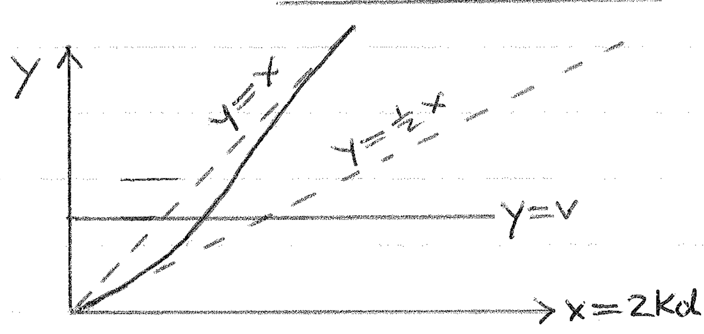


(d) Eliminating A_1 between equations ① and ② from (b)

$$-KB_2(e^{Kd} + \cancel{e^{-Kd}}) = KB_2(e^{Kd} - \cancel{e^{-Kd}}) - \frac{v}{d} B_2 (e^{Kd} + e^{-Kd})$$

$$2Ke^{Kd} = \frac{v}{d}(e^{Kd} + e^{-Kd})$$

$$\underline{\underline{\frac{2Kd}{1 + e^{-2Kd}} = v}}$$



$Y = \frac{x}{1+e^{-x}}$ (solid curve) runs from 0 to ∞ as x runs from 0 to ∞

$\Rightarrow$ always intercepts $y = v$

(e) Since $1 \leq 1 + e^{-2Kd} \leq 2$, Eq (1) $\Rightarrow$ $v \leq 2Kd \leq 2v$. Thus, for $v \ll 1$ it is also true that $2Kd \ll 1$ and hence $1 + e^{-2Kd} \simeq 2$.

Going back to Eq.(1), $\frac{2Kd}{2} \approx v$

$$K \approx \frac{v}{d}$$

$$\Rightarrow \quad E_0 = -\frac{\hbar^2 K^2}{2m} \approx -\frac{\hbar^2 v^2}{2md^2}$$

2(a) $$H_{mn} = \langle m|\hat{H}|n\rangle = \begin{pmatrix} 0 & -2i\epsilon \\ 2i\epsilon & 3\epsilon \end{pmatrix}_{mn}$$

$$A_{mn} = \langle m|\hat{A}|n\rangle = \begin{pmatrix} \alpha & 0 \\ 0 & 3\alpha \end{pmatrix}_{mn}$$

(b) Characteristic equation: $0 = |\hat{H} - E\hat{I}| = \begin{vmatrix} -E & -2i\epsilon \\ 2i\epsilon & 3\epsilon - E \end{vmatrix}$

$$= (-E)(3\epsilon - E) - (2i\epsilon)(-2i\epsilon)$$

$$= E^2 - 3\epsilon E - 4\epsilon^2$$

$$E = \frac{3\epsilon}{2} \pm \frac{1}{2}\sqrt{9\epsilon^2 + 16\epsilon^2} = \left(\frac{3}{2} \pm \frac{5}{2}\right)\epsilon$$

$$E_1 = -\epsilon, \quad E_2 = 4\epsilon$$

$E_1 = -\epsilon$: $$\begin{pmatrix} \epsilon & -2i\epsilon \\ 2i\epsilon & 4\epsilon \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{e^{i\theta_1}}{\sqrt{5}}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

Want $e^{i\theta_1} \times 2i$ real and positive so choose $\theta_1 = \frac{3\pi}{2}$.

$$\Rightarrow \quad |E_1\rangle = \frac{1}{\sqrt{5}}(2|1\rangle - i|2\rangle)$$

By orthonormality, $$|E_2\rangle = \frac{1}{\sqrt{5}}(|1\rangle + 2i|2\rangle)$$

(c) $$|\psi(0)\rangle = \frac{1}{\sqrt{5}}(2|1\rangle - i|2\rangle) = |E_1\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = e^{-iE_1 t/\hbar}|E_1\rangle = e^{i\epsilon t/\hbar}\frac{1}{\sqrt{5}}(2|1\rangle - i|2\rangle)$$

(d) A measurement of A will yield one of the eigenvalues of $\hat{A}$. Since A_{mn} is diagonal in the basis $\{|1\rangle, |2\rangle\}$, $|1\rangle$ and $|2\rangle$ are eigenvectors of $\hat{A}$ with eigenvalues $a = \alpha$ and $a = 3\alpha$, respectively.

$$\text{Probability}(A=\alpha) = |\langle\alpha|\psi(t)\rangle|^2 = |\langle 1|\psi(t)\rangle|^2$$

$$= \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5}$$

$$\text{Probability}(A=3\alpha) = |\langle 3\alpha|\psi(t)\rangle|^2 = |\langle 2|\psi(t)\rangle|^2$$

$$= \left|-\frac{i}{\sqrt{5}}\right|^2 = \frac{1}{5}$$