PHY 4604 Fall 2010 - Final Exam

1(a) For a particle of mass m in a 1D box of length L, the energies are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \propto n^2 \qquad \text{for } n = 1, 2, 3, \ldots$$

$$\Rightarrow \qquad E_3 : E_1 = 9:1 \qquad \underline{\text{Answer D}}$$

(b) Radial prob. density $\rho(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\phi \, |\psi(r,\theta,\phi)|^2 r^2 \sin\theta$

$$\xrightarrow{\psi \text{ ind. of } \theta, \phi} 4\pi r^2 |\psi|^2 \qquad \underline{\text{Answer D}}$$

2(a) $\psi(r,\theta,\phi) = A(i r\sin\theta\sin\phi - 7r\sin\theta\cos\phi)\, e^{-r/2a_0}$

$$= \underline{A r e^{-r/2a_0} \sin\theta (i\sin\phi - 7\cos\phi)}$$

(b) Can write $\psi = R(r)\, Y(\theta,\phi)$

where $R(r) = Bre^{-r/2a_0}$ and $Y(\theta,\phi)$ are normalized separately.

Then $\langle r^\nu \rangle = \dfrac{\int_0^\infty r^\nu\, r^2 |R(r)|^2 dr}{\int_0^\infty r^2 |R(r)|^2 dr} = \dfrac{|B|^2 \int_0^\infty r^{\nu+4} e^{-r/a_0} dr}{|B|^2 \int_0^\infty r^4 e^{-r/a_0} dr}$

$$= a_0^\nu \frac{\int_0^\infty u^{\nu+4} e^{-u} du}{\int_0^\infty u^4 e^{-u} du} = \frac{(\nu+4)!}{4!} a_0^\nu \quad \text{using integral provided}$$

We want $\sigma_r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$

$$= \sqrt{\frac{6!}{4!} a_0^2 - \left(\frac{5!}{4!} a_0\right)^2} = \sqrt{30a_0^2 - 25a_0^2}$$

$$= \underline{\sqrt{5}\, a_0}$$

(c) From formula sheet, $r e^{-r/2a_0} = \sqrt{24}\, a_0^{5/2} R_{21}(r)$.

Also, $\sin\theta(i\sin\phi - 7\cos\phi) = \frac{1}{2}\sin\theta(e^{i\phi} - e^{-i\phi} - 7e^{i\phi} - 7e^{-i\phi})$

$$= -\sin\theta(3e^{i\phi} + 4e^{-i\phi})$$

$$= 3\sqrt{\frac{8\pi}{3}} Y_1^1(\theta,\phi) - 4\sqrt{\frac{8\pi}{3}} Y_1^{-1}(\theta,\phi)$$

$$\Rightarrow \qquad \psi(r,\theta,\phi) = 8A\sqrt{\pi}\, a_0^{5/2} R_{21}(r)\left[3Y_1^1(\theta,\phi) - 4Y_1^{-1}(\theta,\phi)\right]$$

$$= \underline{R_{21}(r)\left[\frac{3}{5} Y_1^1(\theta,\phi) - \frac{4}{5} Y_1^{-1}(\theta,\phi)\right]}$$

using the orthonormality of the R_{nl}'s and the Y_l^m's.

2(d) Since $\hat{L}_z Y_l^m = m\hbar Y_l^m$, the possible results are

$L_z = \hbar$ with probability $|c_{211}|^2 = \left|\frac{3}{5}\right|^2$ $\qquad P(L_z = \hbar) = \frac{9}{25}$

$L_z = -\hbar$ " " $|c_{21-1}|^2 = \left|-\frac{4}{5}\right|^2$ $\qquad P(L_z = -\hbar) = \frac{16}{25}$

3(a) The eigenstates of $\hat{S}_y$ are $|S_y = \pm\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle)$.

Thus, the possible results of a measurement of S_y are

$S_y = \frac{\hbar}{2}$ with probability $|\langle S_y = \frac{\hbar}{2}|\chi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\cdot\frac{4}{5} + \frac{-i}{\sqrt{2}}\cdot\frac{-3i}{5}\right|^2$ $\quad P(\frac{\hbar}{2}) = \frac{1}{50}$

$S_y = -\frac{\hbar}{2}$ " " $|\langle S_y = -\frac{\hbar}{2}|\chi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\cdot\frac{4}{5} + \frac{i}{\sqrt{2}}\cdot\frac{-3i}{5}\right|^2$ $\quad P(-\frac{\hbar}{2}) = \frac{49}{50}$

(b) The measurement yields $S_y = \frac{\hbar}{2}$ and leaves the electron in spin state

$$|\chi(0)\rangle = e^{i\alpha}|S_y = \tfrac{\hbar}{2}\rangle = \frac{e^{i\alpha}}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle), \qquad \alpha = \text{real}$$

In a magnetic field along z the stationary states are $|\uparrow\rangle$ with $E_\uparrow = -\gamma B\frac{\hbar}{2}$ and $|\downarrow\rangle$ with $E_\downarrow = \gamma B\frac{\hbar}{2}$. Thus the propagator is

$$U(t) = e^{-iE_\uparrow t/\hbar}|\uparrow\rangle\langle\uparrow| + e^{-iE_\downarrow t/\hbar}|\downarrow\rangle\langle\downarrow|$$

and $$|\chi(t)\rangle = U(t)|\chi(0)\rangle = \frac{e^{i\alpha}}{\sqrt{2}}\left(e^{i\gamma Bt/2}|\uparrow\rangle + i\,e^{-i\gamma Bt/2}|\downarrow\rangle\right)$$

(c)
$$\langle S_y(t)\rangle = \langle\chi(t)|\hat{S}_y|\chi(t)\rangle$$
$$= \frac{e^{-i\alpha}}{\sqrt{2}}\left(e^{-i\gamma Bt/2} \;\; -ie^{i\gamma Bt/2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{e^{i\alpha}}{\sqrt{2}}\begin{pmatrix}e^{i\gamma Bt/2}\\ ie^{-i\gamma Bt/2}\end{pmatrix}$$
$$= \frac{\hbar}{2}\cos(\gamma Bt)$$

Since $\quad S_y^2 \longleftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\Rightarrow \quad \langle S_y^2(t)\rangle = \langle\chi(t)|\frac{\hbar^2}{4}\hat{I}|\chi(t)\rangle$
$$= \frac{\hbar^2}{4}$$

Uncertainty $\quad \sigma_{S_y} = \sqrt{\langle S_y^2(t)\rangle - \langle S_y(t)\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left[\frac{\hbar}{2}\cos(\gamma Bt)\right]^2}$
$$= \frac{\hbar}{2}\sin(\gamma Bt)$$

The uncertainty vanishes at $t = \frac{\pi}{\gamma B}$, at which time

$$\langle S_x\rangle = \frac{e^{-i\alpha}}{\sqrt{2}}(-i \;\; 1)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{e^{i\alpha}}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix}$$
$$= 0$$