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The Gaussian wave packet is of particular importance, and is one of the few examples that can be analyzed algebraically (Griffiths Prob. 2.22).

1) Initial state: $\psi(x,0) = (2\pi W^2)^{-1/4} e^{i p_0 x/\hbar - (x/2W)^2}$ (4)

By symmetry $\langle x \rangle = 0$

while $\langle x^2 \rangle = \frac{1}{\sqrt{2\pi}\, W} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2W^2}$

Useful integrals (see Shankar A.2):

(a) Let $I_n(\alpha) = \int_{-\infty}^{+\infty} x^n e^{-\alpha x^2} dx$

for $\alpha$ pure real or pure imaginary.

Then $I_0(\alpha) = \sqrt{\pi/\alpha}$

$I_{2n+1}(\alpha) = 0$ (odd integrand)

$I_{2n}(\alpha) = \left(-\frac{d}{d\alpha}\right)^n I_0(\alpha)$

$\Rightarrow$ $I_2(\alpha) = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$.

(b) For complex $\alpha$ and $\beta$ such that $\mathrm{Re}\,\alpha > 0$

$I_0(\alpha,\beta) = \int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\pi/\alpha}\; e^{\beta^2/4\alpha}$.

Using $I_2$ with $\alpha = \frac{1}{2w^2}$,

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi}\, w} w^2 \sqrt{2\pi w^2}$$
$$= w^2$$
$$\Rightarrow \quad \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$
$$= w$$

Also
$$\hat{p}\,\psi(x,0) = \left(p_0 + \frac{i\hbar x}{2w^2}\right)\psi(x,0)$$
$$\Rightarrow \quad \langle p \rangle = \int_{-\infty}^{+\infty} \left(p_0 + \underbrace{\frac{i\hbar x}{2w^2}}_{\text{odd}}\right) \underbrace{|\psi(x,0)|^2}_{\text{even}}$$
$$= p_0$$
$$\hat{p}^2\,\psi(x,0) = \left[\left(p_0 + \frac{i\hbar x}{2w^2}\right)^2 + \frac{\hbar^2}{2w^2}\right]\psi(x,0)$$
$$\Rightarrow \quad \langle p^2 \rangle = \int_{-\infty}^{+\infty} \left(p_0^2 + \frac{i\hbar x}{w^2} - \frac{\hbar^2 x^2}{4w^4} + \frac{\hbar^2}{2w^2}\right)|\psi(x,0)|^2$$
$$= p_0^2 + \frac{\hbar^2}{2w^2} - \frac{\hbar^2}{4w^4}\langle x^2 \rangle$$
$$= p_0^2 + \frac{\hbar^2}{4w^2} \quad = 2m\langle E \rangle \text{ since } V = 0$$
$$\Rightarrow \quad \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$
$$= \frac{\hbar}{2w}$$

Note that $\sigma_x \sigma_p = \frac{\hbar}{2}$

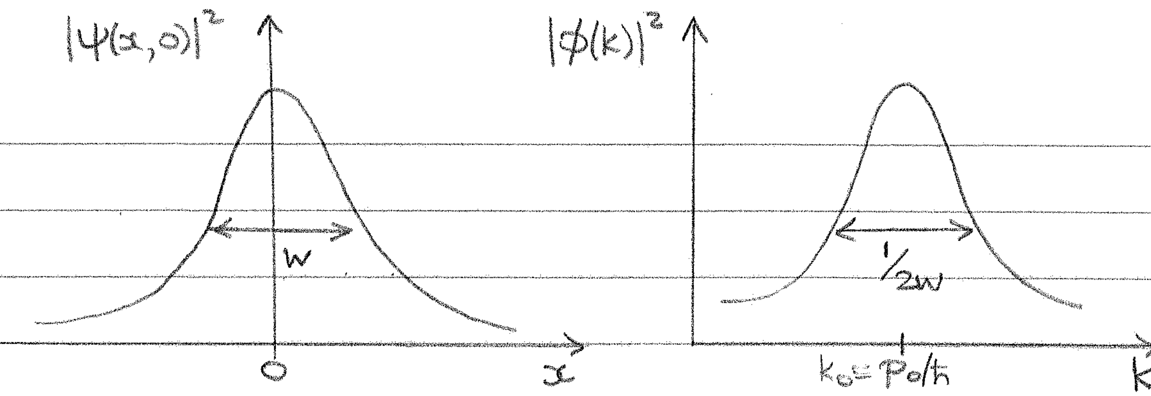
so $\psi(x,0)$ is a minimum-uncertainty state.

Note also that can create a wave packet centered on $x = x_0$ by replacing $x$ by $x - x_0$ on the RHS of Eq. (4)

2) Decomposition into stationary states:

Eq. (2) $\Rightarrow$
$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x,0)\, e^{-ikx}\, dx$$
$$= \frac{1}{(2\pi)^{3/4}\sqrt{w}} \int_{-\infty}^{+\infty} e^{-x^2/4w^2 + i(p_0/\hbar - k)x}\, dx$$

Using $I_0(\alpha, \beta)$ with $\alpha = \frac{1}{4w^2}$ and $\beta = i(p_0/\hbar - k)$,
$$\phi(k) = \frac{1}{(2\pi)^{3/4}\sqrt{w}} \sqrt{\pi}\, 2w\, e^{-w^2(k - p_0/\hbar)^2}$$
$$= (2w^2/\pi)^{1/4}\, e^{-w^2(k - p_0/\hbar)^2}$$




There is an inverse relation between the width of the peaks in $|\psi(x,0)|^2$ and $|\phi(k)|^2$, which is typical of Fourier transforms.

3) Time-evolution of the wave packet:

Eq. (3) $\Rightarrow$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i(kx - \hbar k^2 t/2m)}\, dk$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2w^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-w^2(k-p_0/\hbar)^2 + i(kx - \hbar k^2 t/2m)}\, dk$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2w^2}{\pi}\right)^{1/4} e^{-(wp_0/\hbar)^2}\, I_0\!\left(w^2 + \frac{i\hbar t}{2m},\ \frac{2w^2 p_0}{\hbar} + ix\right)$$

After some tedious algebra, the result can be cast in the form

$$\psi(x,t) = (2\pi w_t^2)^{-1/4}\, e^{ip_0(x - v_p t)/\hbar - (x - v_g t)^2/(2w_t)^2}$$

where

$$v_p = \frac{p_0}{2m}$$
$$v_g = \frac{p_0}{m}$$
$$w_t = w\sqrt{1 + \frac{it}{\tau}}$$
$$\tau = \frac{2mw^2}{\hbar}$$

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Here,

$$v_p = \frac{E_k}{\hbar k}\Big|_{k=k_0}$$

is the phase velocity, which determines the speed at which points of constant phase travel. This must be distinguished from the group velocity

$$v_g = \frac{1}{\hbar} \frac{dE_k}{dk}\Big|_{k=k_0},$$

which determines the speed of the peak in the envelope formed by the superposition of waves traveling near the phase velocity. Here we have the special case $\nu = 2$ of the result from wave propagation that if the angular frequency goes as $\omega \propto k^\nu$, then $v_g = \nu v_p$.

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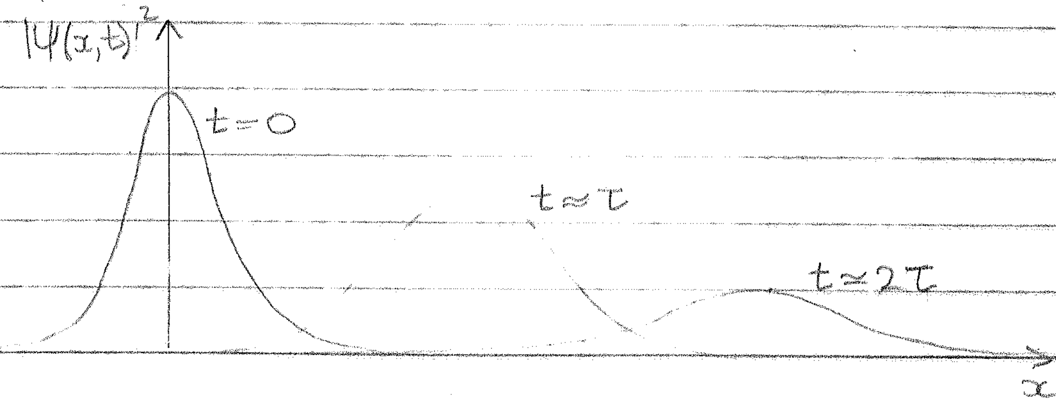
If we calculate expectation values using similar methods to those for $t=0$, we find

$$\langle x\rangle = v_g t = \frac{\langle p\rangle t}{m} \qquad \sigma_x = w\sqrt{1+(t/\tau)^2}$$

$$\langle p\rangle = p_0 \qquad \sigma_p = \frac{\hbar}{2w}$$




Note that

- $\langle x\rangle$ and $\langle p\rangle$ exactly obey the classical equations of motion for a free particle;
- $\sigma_p$ is constant but $\sigma_x$ increases monotonically, initially quadratically in time (for $t \ll \tau$) but eventually linearly (for $t \gg \tau$);
- $\tau = \frac{2mw^2}{\hbar}$ can vary enormously depending on $m$ and $w$:

  nanoscopic: $m = 9\times10^{-31}$ kg (electron), $w = 10^{-9}$ m $\Rightarrow \tau = 2\times10^{-14}$ s.

  microscopic $m = 10^{-9}$ kg (1 μg), $w = 10^{-6}$ m (1 μm) $\Rightarrow \tau = 6\times10^{5}$ yr

  macroscopic $m = 10^{-3}$ kg (1 g), $w = 0.1$ m $\Rightarrow \tau = 6\times10^{22}$ yr

  $\Rightarrow$ rate of wave packet spreading is negligible above the atomic scale