PHY 4604 Spring 2012 - Exam 1

[20] 1(a) Note that $\Psi(x,t_0) = \Psi(-x,t_0)$.

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{+\infty} \underbrace{\Psi^*(x,t_0)}_{\text{even}}\ \underbrace{x}_{\text{odd}}\ \underbrace{\Psi(x,t_0)}_{\text{even}}\,dx \\ &= 0 \qquad \text{by symmetry}\end{aligned}$$

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{+\infty} \Psi^*(x,t_0)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t_0)\,dx \\ &= -i\hbar\int_{-\infty}^{+\infty} \underbrace{\Psi^*(x,t_0)}_{\text{even}}\ \underbrace{\frac{\partial\Psi(x,t_0)}{dx}}_{\text{odd}}\,dx \\ &= 0 \qquad \text{by symmetry}\end{aligned}$$

Note also that $\Psi(x,t_0) = \frac{3}{5}\psi_1(x) + \frac{4}{5}\psi_3(x)$

where $\hat{H}\psi_n(x) = \frac{n^2\pi^2\hbar^2}{2ma^2}\psi_n(x)$

$$\Rightarrow \quad \begin{aligned}\langle H\rangle &= \int_{-\infty}^{+\infty} \Psi^*(x,t_0)\,\hat{H}\,\Psi(x,t_0)\,dx \\ &= \left(\frac{3}{5}\right)^2\cdot\frac{\pi^2\hbar^2}{2ma^2} + \left(\frac{4}{5}\right)^2\frac{3^2\pi^2\hbar^2}{2ma^2} \\ &= \frac{153\pi^2\hbar^2}{50ma^2}\end{aligned}$$

[20] (b)

$$\begin{aligned}\text{prob.}\left(-\tfrac{a}{4}<x<\tfrac{a}{4}\right) &= \int_{-a/4}^{a/4} |\Psi(x,t_0)|^2\,dx \\ &= \frac{2}{25a}\int_{-a/4}^{a/4}\left[9\cos^2\frac{\pi x}{a} + 16\cos^2\frac{3\pi x}{a} + 24\cos\frac{\pi x}{a}\cos\frac{3\pi x}{a}\right]dx \\ &= \frac{2}{25a}\left[\frac{9a}{\pi}\int_{-\pi/4}^{\pi/4}\cos^2 y\,dy + \frac{16a}{3\pi}\int_{-3\pi/4}^{3\pi/4}\cos^2 y\,dy \right.\\ &\qquad \left. + 12\int_{-a/4}^{a/4}\left(\cos\frac{4\pi x}{a} + \cos\frac{2\pi x}{a}\right)dx\right] \\ &= \frac{2}{25a}\left\{\frac{9a}{\pi}\left[\frac{y}{2} + \frac{\sin 2y}{4}\right]_{-\pi/4}^{\pi/4} + \frac{16a}{3\pi}\left[\frac{y}{2} + \frac{\sin 2y}{4}\right]_{-3\pi/4}^{3\pi/4} \right.\\ &\qquad \left. + 12\left[\frac{a}{4\pi}\sin\frac{4\pi x}{a} + \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_{-a/4}^{a/4}\right\} \\ &= \frac{2}{25}\left[\frac{9}{\pi}\left(\frac{\pi}{4}+\frac{1}{2}\right) + \frac{16}{3\pi}\left(\frac{3\pi}{4}-\frac{1}{2}\right) + 12\left(0+\frac{1}{2\pi}\cdot 2\right)\right] \\ &= \frac{1}{2} + \frac{83}{75\pi}\end{aligned}$$

The classical probability density is proportional to $\frac{1}{\text{speed}}$, which is uniform over the entire region $-a/2 < x < a/2$, so

$$\text{prob}\left(-\tfrac{a}{4}<x<\tfrac{a}{4}\right) = \int_{-a/4}^{a/4}\frac{1}{a}\,dx = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\text{quantum probability}}{\text{classical probability}} = 1 + \frac{166}{75\pi}$$

[7] 2(a)

$$\Psi(x,t) = \cos\theta\, \psi_1(x)\, e^{-iE_1 t/\hbar} + e^{i\phi}\sin\theta\, \psi_2(x)\, e^{-iE_2 t/\hbar}$$

where $E_n = (n+\frac{1}{2})\hbar\omega$

[18] (b)

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t) \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)\, \Psi(x,t)\, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{+\infty} \left(\cos\theta\, \psi_1^* e^{iE_1 t/\hbar} + e^{-i\phi}\sin\theta\, \psi_2^* e^{iE_2 t/\hbar}\right)$$

$$\times\left[\cos\theta(\sqrt{2}\psi_2 + \sqrt{1}\psi_0)e^{-iE_1 t/\hbar} + e^{i\phi}\sin\theta\,(\sqrt{3}\psi_3 + \sqrt{2}\psi_1)e^{-iE_2 t/\hbar}\right] dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{2}e^{i\phi}\cos\theta\sin\theta\, e^{i(E_1-E_2)t/\hbar} + \sqrt{2}e^{-i\phi}\cos\theta\sin\theta\, e^{i(E_2-E_1)t/\hbar}\right]$$

using $\int_{-\infty}^{+\infty} \psi_m^* \psi_n\, dx = \delta_{m,n}$.

$$\Rightarrow \quad \langle x \rangle = \sqrt{\frac{\hbar}{m\omega}}\, 2\cos\theta\sin\theta\, \mathrm{Re}\left[e^{i(\omega t-\phi)}\right]$$

$$= \sqrt{\frac{\hbar}{m\omega}} \sin 2\theta \cos(\omega t - \phi).$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\; i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)\, \Psi(x,t)\, dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{+\infty} \left(\cos\theta\, \psi_1^* e^{iE_1 t/\hbar} + e^{-i\phi}\sin\theta\, \psi_2^* e^{iE_2 t/\hbar}\right)$$

$$\times\left[\cos\theta(\sqrt{2}\psi_2 - \sqrt{1}\psi_0)e^{-iE_1 t/\hbar} + e^{i\phi}\sin\theta(\sqrt{3}\psi_3 - \sqrt{2}\psi_1)e^{-iE_2 t/\hbar}\right] dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[-\sqrt{2}e^{i\phi}\cos\theta\sin\theta\, e^{i(E_1-E_2)t/\hbar} + \sqrt{2}e^{-i\phi}\cos\theta\sin\theta\, e^{i(E_2-E_1)t/\hbar}\right]$$

$$= -\sqrt{\hbar m\omega}\, 2\cos\theta\sin\theta\, \mathrm{Im}\left[e^{i(\omega t-\phi)}\right]$$

$$= -\sqrt{\hbar m\omega} \sin 2\theta \sin(\omega t - \phi)$$

[5] (c) To maximize the amplitude of the oscillation of $\langle x \rangle$, we want

$$|\sin 2\theta| = 1$$

$$\theta = (k+\tfrac{1}{2})\frac{\pi}{2} \qquad \text{for any integer } k.$$

[5] (d) To make $\langle x \rangle$ vanish at $t=0$ we want

$$\cos(-\phi) = 0$$

$$\phi = (k+\tfrac{1}{2})\pi \qquad \text{for any integer } k.$$

[5] 3(a) We know that ψ_a describes the $n=4$ state of the infinite square well:

$$\psi_a(x) = \sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L}.$$

[20] (b) (i) The third excited bound state of any potential has 3 nodes.

(ii) ψ_b must satisfy $\frac{1}{\psi_b}\frac{d^2\psi_b}{dx^2} = V(x) - E_b$

which presumably is negative for all $0 < x < L$, but most negative near $x = 0$ and least negative near $x = L$.

$\Rightarrow$ The "curvature" $\left|\frac{1}{\psi_b}\frac{d^2\psi_b}{dx^2}\right|$ decreases as x increases across the well, and so the nodes must be shifted leftward from their positions for $V_0 = 0$.

(iii) Classically, the probability density

$$P(x) = \frac{A}{v(x)} = A\sqrt{\frac{m}{2[E - V(x)]}}$$

which increases as $V(x)$ rises to approach E from below.

$\Rightarrow$ By analogy, we expect a greater quantum mechanical probability density in the right half of the well than in the left.

$\Rightarrow$ We expect the maximum of $|\psi_b(x)|$ over $0 < x < L/2$ to be less than the maximum of $|\psi_b(x)|$ over $L/2 < x < L$.

Based on the above, $\psi_b(x)$ should look something like the following: