PHY 4604 Spring 2012 — Exam 2

1(a) $$\hat{H} = \sum_{j,k} H_{jk} |j\rangle\langle k| = \epsilon\left[|1\rangle\langle 1| + |2\rangle\langle 2| + i\left(|2\rangle\langle 1| - |1\rangle\langle 2|\right)\right]$$

$$\hat{C} = \sum_{j,k} C_{jk} |j\rangle\langle k| = \gamma\left[4\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) + 3\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)\right]$$

(b) The characteristic equation for the eigenvalues E of $\hat{H}$ is

$$0 = |\hat{H} - E\hat{I}| = \begin{vmatrix} \epsilon - E & -i\epsilon \\ i\epsilon & \epsilon - E \end{vmatrix} = (\epsilon - E)^2 - E^2 = E(E - 2\epsilon)$$

$\underline{\underline{E_1 = 0}}$: $$\begin{pmatrix} \epsilon & -i\epsilon \\ i\epsilon & \epsilon \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ or } |E_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - i|2\rangle\right)$$

$\underline{\underline{E_2 = 2\epsilon}}$: $$\begin{pmatrix} -\epsilon & -i\epsilon \\ i\epsilon & -\epsilon \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ or } |E_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right)$$

(c) We can invert the above equations for $|E_1\rangle$, $|E_2\rangle$ to obtain

$$|\psi(0)\rangle = |2\rangle = \frac{i}{\sqrt{2}}\left(|E_1\rangle - |E_2\rangle\right)$$

Each energy eigenket evolves independently in time, so

$$\begin{aligned}
|\psi(t)\rangle &= \frac{i}{\sqrt{2}}\left(e^{-iE_1 t/\hbar}|E_1\rangle - e^{-iE_2 t/\hbar}|E_2\rangle\right) \\
&= \frac{i}{\sqrt{2}}\left[1\,\frac{1}{\sqrt{2}}\left(|1\rangle - i|2\rangle\right) - e^{-2i\epsilon t/\hbar}\frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right)\right] \\
&= \frac{i}{2}\left(1 - e^{-2i\epsilon t/\hbar}\right)|1\rangle + \frac{1}{2}\left(1 + e^{-2i\epsilon t/\hbar}\right)|2\rangle \\
&= e^{-i\epsilon t/\hbar}\left(\cos\frac{\epsilon t}{\hbar}|2\rangle - \sin\frac{\epsilon t}{\hbar}|1\rangle\right)
\end{aligned}$$

(d) The only possible results of a measurement of C are the eigenvalues of $\hat{C}$, which satisfy the characteristic equation

$$0 = |\hat{C} - c\hat{I}| = \begin{vmatrix} 4\gamma - c & 3\gamma \\ 3\gamma & -4\gamma - c \end{vmatrix} = c^2 - 16\gamma^2 - 9\gamma = (c + 5\gamma)(c - 5\gamma)$$

$\underline{\underline{c_1 = -5\gamma}}$: $$\begin{pmatrix} 9\gamma & 3\gamma \\ 3\gamma & \gamma \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \text{ or } |c_1\rangle = \frac{1}{\sqrt{10}}\left(|1\rangle - 3|2\rangle\right)$$

$\underline{\underline{c_2 = 5\gamma}}$: $$\begin{pmatrix} -\gamma & 3\gamma \\ 3\gamma & -9\gamma \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ or } |c_2\rangle = \frac{1}{\sqrt{10}}\left(3|1\rangle + |2\rangle\right)$$

The probability of obtaining $C = c_j$ is $P(C = c_j) = |\langle c_j|\psi\rangle|^2$.

Here, can get $C = C_1 = -5\gamma$ with prob. $P(C_1) = |\langle C_1|2\rangle|^2 = \frac{9}{10}$

or $C = C_2 = 5\gamma$ with prob. $P(C_2) = |\langle C_2|2\rangle|^2 = \frac{1}{10}$

2(a) Since $V(x) = \infty$ for all $|x| > \frac{a}{2}$, require $\psi(\pm\frac{a}{2}) = 0$.

At the location $x = 0$ of the delta function in $V(x)$ require

$\lim_{\varepsilon\to 0^+} \psi(\varepsilon) - \psi(-\varepsilon) = 0$ and $\lim_{\varepsilon\to 0^+} \psi'(\varepsilon) - \psi'(-\varepsilon) = \frac{2m}{\hbar^2} V_0 a\, \psi(0)$.

(b) The standard particle-in-a-box solutions are

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\frac{n\pi x}{a} & \text{for } n = 1,3,5,\ldots \\ \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} & \text{for } n = 2,4,6,\ldots \end{cases} \quad \text{with } E_n(0) = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

(c) The odd-parity ψ_n's vanish at $x = 0$ and so are not affected by the delta function. They still solve the TISE with $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, for n = even.

(d) Even-parity solutions that are consistent with $V(x) = 0$ for all $0 < |x| < \frac{a}{2}$ and vanish at $|x| = \frac{a}{2}$ can be written in the form

$$\psi_n(x) = \begin{cases} A_n \sin k_n\left(\frac{a}{2} - |x|\right) & \text{for } |x| \leq \frac{a}{2} \\ 0 & \text{for } |x| \geq \frac{a}{2} \end{cases}$$

The allowed k_n's are determined by the second b.c. at $x = 0$:

$$-2k_n A_n \cos\frac{k_n a}{2} = \frac{2m}{\hbar^2} V_0 a A_n \sin\frac{k_n a}{2}$$

$$\Rightarrow \quad \tan\frac{k_n a}{2} = -\frac{\hbar^2 k_n}{mV_0 a}.$$

(e) $$\frac{k_n a}{2} = \operatorname{atan}\left(-\frac{\hbar^2 k_n}{mV_0 a}\right) \simeq \frac{n\pi}{2} + \frac{mV_0 a}{\hbar^2 k_n}$$ {Using the formula given and the fact that n is an odd integer

$$\Rightarrow \quad k_n \simeq \underbrace{\frac{n\pi}{a}}_{\text{bigger}} + \underbrace{\frac{2mV_0}{\hbar^2 k_n}}_{\text{smaller}} \simeq \frac{n\pi}{a} + \frac{2mV_0}{\hbar^2(n\pi/a)}$$ {putting $k_n \simeq n\pi/a$ in the smaller term

$$\simeq \frac{n\pi}{a} + \frac{2mV_0 a}{n\pi\hbar^2}$$

Then $$E_n(V_0) = \frac{\hbar^2 k_n^2}{2m} \simeq \frac{n^2\pi^2\hbar^2}{2ma^2} + 2V_0$$

$$\Rightarrow \quad E_n(V_0) - E_n(0) \simeq 2V_0 \quad \text{for } n = 1,3,5,\ldots$$

$$= 0 \quad \text{for } n = 2,4,6,\ldots$$