PHY 4604 Spring 2012 – Final Exam

[4] 1(a) A zero-angular-momentum wave function is of the form

$$\psi(r,\theta,\phi) = R(r)\,Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\,R(r).$$

The wave function has no dependence on θ or ϕ. It vanishes only on the surfaces of spheres of radius r_0 where $R(r_0)=0$.

$\Rightarrow$ Only statement B is true.

[6] (b) A Hermitian operator is defined by the property $\hat{A}^\dagger \equiv (A^*)^T = \hat{A}$, or equivalently, $\hat{A}^T = \hat{A}^*$. Such an operator must have real eigenvalues, but it is not required that $\hat{A}^\dagger\hat{A} = \hat{I}$ or $\langle \hat{A}\rangle \geq 0$.

$\Rightarrow$ Statements A, D, E are true.

2. We have $\hat{S}_j \leftrightarrow \frac{\hbar}{2}\sigma_j$ with σ_j a Pauli matrix
and $\hat{S}_j^2 \leftrightarrow \left(\frac{\hbar}{2}\right)^2 \sigma_j^2 = \left(\frac{\hbar}{2}\right)^2 I$
} for $j = x, y, z$

[15] (a) i. The possible results of an S_z measurement are the eigenvalues $\pm\frac{\hbar}{2}$ of $\hat{S}_z$.

$$\text{prob}(S_z = +\tfrac{\hbar}{2}) = |\langle S_z = +\tfrac{\hbar}{2} | \chi_0\rangle|^2 = \left|(1\ \ 0)\cdot\tfrac{1}{5}\begin{pmatrix}4i\\3\end{pmatrix}\right|^2 = \frac{16}{25}$$

$$\text{prob}(S_z = -\tfrac{\hbar}{2}) = |\langle S_z = -\tfrac{\hbar}{2} | \chi_0\rangle|^2 = \left|(0\ \ 1)\cdot\tfrac{1}{5}\begin{pmatrix}4i\\3\end{pmatrix}\right|^2 = \frac{9}{25}$$

ii.
$$\langle S_z\rangle = \frac{16}{25}\left(\frac{\hbar}{2}\right) + \frac{9}{25}\left(-\frac{\hbar}{2}\right) = \frac{7}{25}\,\frac{\hbar}{2}$$

iii.
$$\sigma_{S_z} = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{1-\left(\frac{7}{25}\right)^2}\;\frac{\hbar}{2} = \frac{24}{25}\,\frac{\hbar}{2}$$

[18] (b) After the first measurement, the wave function will be the $S_z = +\frac{\hbar}{2}$ eigenstate:

$$\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$$

i. The possible results of an S_x measurement are the eigenvalues $\pm\frac{\hbar}{2}$ of $\hat{S}_x$.

$$\text{prob}(S_x = +\tfrac{\hbar}{2}) = |\langle S_x = +\tfrac{\hbar}{2} | \chi_1\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}(1\ \ 1)\cdot\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\text{prob}(S_x = -\tfrac{\hbar}{2}) = |\langle S_x = -\tfrac{\hbar}{2} | \chi_1\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}(1\ \ -1)\cdot\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

ii. $\langle S_x \rangle = \frac{1}{2}\left(\frac{\hbar}{2}\right) + \frac{1}{2}\left(-\frac{\hbar}{2}\right) = 0$

iii. $\sigma_{S_x} = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{1-0}\,\frac{\hbar}{2} = \frac{\hbar}{2}$

[12] (c) i. The possible results are again the eigenvalues $\pm\frac{\hbar}{2}$ of $\hat{S}_x$.

$$\text{prob}\left(S_x = +\tfrac{\hbar}{2}\right) = \left|\langle S_x = +\tfrac{\hbar}{2} | \chi_0 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}(1 \;\; 1)\cdot \tfrac{1}{5}\begin{pmatrix}4i\\3\end{pmatrix}\right|^2 = \tfrac{1}{2}$$

$$\text{prob}\left(S_x = -\tfrac{\hbar}{2}\right) = \left|\langle S_x = -\tfrac{\hbar}{2} | \chi_0 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}(1 \;\; -1)\cdot \tfrac{1}{5}\begin{pmatrix}4i\\3\end{pmatrix}\right|^2 = \tfrac{1}{2}$$

Since the probabilities are the same as in (b), so are the answers to ii and iii.

3(a) We can rewrite the given wave function

$$\psi(r,\theta,\phi) = 2Aa_0\left[\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0} - \frac{r}{a_0}e^{-r/2a_0}\cdot\cos\theta\right]$$

$$= 2Aa_0\left[\sqrt{2a_0^3}\,R_{20}\cdot\sqrt{4\pi}\,Y_0^0 - \sqrt{24a_0^3}\,R_{21}\sqrt{\frac{4\pi}{3}}\,Y_1^0\right]$$

$$= A(2a_0)^{5/2}\sqrt{\pi}\left(R_{20}Y_0^0 - 2R_{21}Y_1^0\right)$$

$$= \frac{1}{\sqrt{5}}\left[R_{20}(r)Y_0^0(\theta,\phi) - 2R_{21}(r)Y_1^0(\theta,\phi)\right]$$

since $\sum_{n,l,m}|c_{nlm}|^2 = 1$ is the normalization condition.

(b)

$$\langle L^2 \rangle = \sum_{n,l,m}|c_{nlm}|^2\, l(l+1)\hbar^2$$

$$= \frac{1}{5}\cdot 0 + \frac{4}{5}\cdot 2\hbar^2 = \frac{8}{5}\hbar^2$$

(c) Since $R_{nl}Y_l^m$ is a state with $L_z = m\hbar$, the only possible result for our ψ is $L_z = 0$ with probability $= 1$.

(d)

$$\langle r \rangle = \int r^2dr\int\sin\theta\, d\theta\int d\phi\; \psi^*(r,\theta,\phi)\, r\, \psi(r,\theta,\phi)$$

$$= \int_0^\infty r^3dr\int_0^\pi\sin\theta\, d\theta\int_0^{2\pi}d\phi\left[\frac{1}{5}|R_{20}Y_0^0|^2 + \frac{4}{5}|R_{21}Y_1^0|^2 - \frac{4}{5}\mathrm{Re}\left(R_{20}^*R_{21}Y_0^{0*}Y_1^0\right)\right]$$

integrates to zero due to orthonormality of Y_l^m's

$$= \int_0^\infty r^3dr\left[\frac{1}{5}\frac{1}{2a_0^3}\left(1-\frac{r}{2a_0}\right)^2e^{-r/a_0} + \frac{4}{5}\frac{1}{24a_0^3}\left(\frac{r}{a_0}\right)^2e^{-r/a_0}\right]$$

$$= \frac{a_0}{5}\int_0^\infty\rho^3d\rho\left[\frac{1}{2}\left(1-\rho+\frac{\rho^2}{4}\right)e^{-\rho} + \frac{4}{24}\rho^2e^{-\rho}\right]$$

$$= \frac{a_0}{30}\int_0^\infty d\rho\,\left(3\rho^3 - 3\rho^4 + \frac{7}{4}\rho^5\right)e^{-\rho}$$

$$= \frac{a_0}{30}\left(3\times3! - 3\times4! + \frac{7}{4}\times5!\right) = \frac{26}{5}a_0$$