PHY 4604 Spring 2013 – Exam 1

1(a) $\Psi(x,0) = \sum_n c_n \psi_n(x)$ evolves into $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$.

Here, $E_n = (n+\frac{1}{2})\hbar\omega$ so

$$\Psi(x,T) = \frac{1}{\sqrt{2}}\left[\psi_0(x) e^{-i\omega T/2} - i\,\psi_2(x) e^{-5i\omega T/2}\right].$$

(b) Since $\psi_0(x)$ and $\psi_2(x)$ are both even under $x \to -x$, so is $\Psi(x,T)$.

$$\Rightarrow \quad \langle x \rangle = \int_{-\infty}^{+\infty} \underbrace{x}_{\text{odd}}\, \underbrace{|\Psi(x,T)|^2}_{\text{even}}\, dx = 0$$

Using $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ and $a_\pm \psi_n = \frac{\sqrt{n+1}}{\sqrt{n}}\psi_{n\pm 1}$,

$$\hat{x}^2 \Psi(x,T) = \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2)\Psi(x,T)$$

$$= \frac{\hbar}{2m\omega}\frac{1}{\sqrt{2}}\left[(\sqrt{2}\psi_2 + \psi_0)e^{-i\omega T/2} - i(\sqrt{12}\psi_4 + 5\psi_2 + \sqrt{2}\psi_0)e^{-5i\omega T/2}\right]$$

Then, using $\int_{-\infty}^{+\infty} \psi_m^*(x)\psi_n(x)\,dx = \delta_{m,n}$,

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,T)\,\hat{x}^2\,\Psi(x,T)\,dx$$

$$= \frac{\hbar}{2m\omega}\frac{1}{2}\left(1 + \sqrt{2}i e^{2i\omega T} + 5 - \sqrt{2}i e^{-2i\omega T}\right) = \frac{\hbar}{2m\omega}\left[3 - \sqrt{2}\sin(2\omega T)\right]$$

Finally, $\sigma_x = +\sqrt{\langle x^2\rangle - \langle x\rangle^2} = +\sqrt{\frac{\hbar}{2m\omega}\left[3 - \sqrt{2}\sin(2\omega T)\right]}$

(c) $$\langle E^k \rangle = \sum_n |c_n|^2 E_n^k$$

$$\Rightarrow \quad \langle E \rangle = \frac{1}{2}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{5\hbar\omega}{2} = \frac{3}{2}\hbar\omega$$

$$\langle E^2 \rangle = \frac{1}{2}\left(\frac{\hbar\omega}{2}\right)^2 + \frac{1}{2}\left(\frac{5\hbar\omega}{2}\right)^2 = \frac{13}{4}(\hbar\omega)^2$$

$$\sigma_E = +\sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{13}{4} - \left(\frac{3}{2}\right)^2}\,\hbar\omega = \hbar\omega$$

2(a) Since $E_1 < V(x) = 0$ for $a < x < \infty$, can only have a decaying term

$$\psi_1(x) = B_5 e^{-\kappa_5 x}$$

$$\kappa_5 = +\sqrt{-2mE_1}/\hbar$$

(b) Since $E_1 > V(x) = -V_0$ for $\frac{a}{2} < x < a$, can have two wavelike terms

$$\psi_1(x) = A_4 e^{ik_4 x} + B_4 e^{-ik_4 x}$$

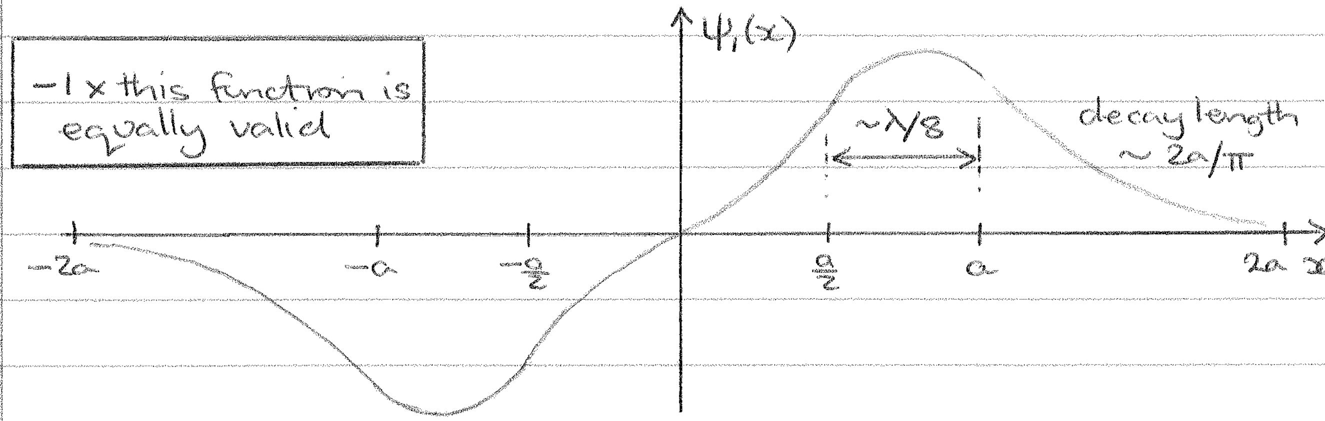
$$k_4 = +\sqrt{2m(E_1 + V_0)}/\hbar$$

(c) Since $E_1 < V(x) = V_0$ for $|x| < a/2$ (a finite region) can have two exponentials:

$$\psi_1(x) = A_3 e^{K_3 x} + B_3 e^{-K_3 x}$$
$$K_3 = +\sqrt{2m(V_0 - E_1)}/\hbar$$

(d) Since $V(-x) = V(x)$, we expect the first excited band state to satisfy $\psi_1(-x) = -\psi_1(x)$ with one node, located at $x = 0$.

For $E_1 \simeq -\frac{\pi^2\hbar^2}{8ma^2}$, $\quad K_5 \simeq K_4 \simeq \frac{\pi}{2a}$, $\quad K_3 \simeq \frac{\sqrt{3}\pi}{2a}$



3(a) can identify $\psi(x) = \frac{1}{\sqrt{5}}\psi_1(x) - \frac{2}{\sqrt{5}}\psi_2(x)$ where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$$\langle E \rangle = \sum_n |c_n|^2 E_n = \left(\frac{1}{5}\cdot 1^2 + \frac{4}{5}\, 2^2\right) E_1 = \frac{17}{10}\frac{\pi^2\hbar^2}{ma^2}$$

(b)
$$\langle E^2 \rangle = \sum_n |c_n|^2 E_n^2 = \left(\frac{1}{5}\cdot 1^2 + \frac{4}{5}\cdot 2^4\right) E_1^2 = \frac{13}{4}\left(\frac{\pi^2\hbar^2}{ma^2}\right)^2$$
$$\sigma_E = +\sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{13}{4} - \left(\frac{17}{10}\right)^2}\ \frac{\pi^2\hbar^2}{ma^2}$$
$$= \frac{3}{5}\frac{\pi^2\hbar^2}{ma^2}$$

(c)
$$P\left(x < \frac{a}{4}\right) = \int_{-\infty}^{a/4} |\psi(x)|^2\, dx$$
$$= \frac{2}{5a}\int_0^{a/4}\left(\sin^2\frac{\pi x}{a} + 4\sin^2\frac{2\pi x}{a} - 4\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right)dx$$
$$\hookrightarrow -8\sin^2\frac{\pi x}{a}\cos\frac{\pi x}{a} = -8\left(1-\cos^2\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}$$
$$= \frac{2}{5a}\left[\frac{a}{\pi}\int_0^{\pi/4}\sin^2 y\, dy + 4\frac{a}{2\pi}\int_0^{\pi/2}\sin^2 y\, dy - 8\frac{a}{\pi}\int_0^{\pi/4}\cos y\, dy + 8\frac{a}{\pi}\int_0^{\pi/4}\cos^3 y\, dy\right]$$
$$= \frac{2}{5\pi}\left[\frac{\pi}{8} - \frac{1}{4} + \frac{4}{2}\left(\frac{\pi}{4} - 0\right) \cancel{- 8\frac{1}{\sqrt{2}}} + 8\left(\cancel{\frac{1}{\sqrt{2}}} - \frac{1}{3}\cdot\frac{1}{2\sqrt{2}}\right)\right]$$
$$= \underbrace{\frac{1}{4}}_{\text{classical}} - \frac{2}{5\pi}\underbrace{\left(\frac{1}{4} + \frac{2\sqrt{2}}{3}\right)}_{\text{quantum correction}} \simeq 0.1$$